CLASSICAL AND NUMERICAL SOLUTIONS OF THE VISCOUS BURGERS EQUATION

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I hereby declare that all information in this document has been obtained and presented in accordance with academic rules and ethical conduct. I also declare that, as required by these rules and conduct, I have fully cited and referenced all material and results that are not original to this work.

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ABSTRACT

CLASSICAL AND NUMERICAL SOLUTIONS OF THE VISCOUS BURGERS EQUATION

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In this study, the viscous Burgers equation is studied both theoretically and numerically. Before introducing our model of interest, we provide a brief historical concept of the equation where we also remind the reader some basic concepts of the partial differential equations. Here the linear advection equation, the solution of which is a particular case of first order quasilinear partial differential equations, is taken into account as an example. The solutions of the inviscid Burgers equation follows by similar process. The main part of this thesis is devoted to the solutions by utilizing the Hopf-Cole transform. In the next part, the model is investigated by implementation of numerical techniques. We used Backward Time Centered Space (BTCS) and Forward Time Centered Space (FTCS) for the numerical results. Comparison of analytical and numerical results and error analysis are described for several examples on the final part of that work.

Keywords: Viscous Burgers equation, Hopf-Cole transformation, Method of Separation of Variables, Finite Difference Method

ÖZ

VİSKOZ BURGER DENKLEMİNİN KLASİK VE NÜMERİK ÇÖZÜMLERİ

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Bu çalışmada viskoz Burgers denklemi hem teorik hem de sayısal olarak incelenmiştir. İlgilendiğimiz modeli tanıtmadan önce, okuyucuya kısmi diferansiyel denklemlerin bazı temel kavramlarını da hatırlattığımız denklemin kısa bir tarihsel bilgisini sağlıyoruz. Burada, birinci dereceden hemen hemen lineer kısmi diferansiyel denklemlerin özel bir durumu olan lineer adveksiyon denklemi bir örnek olarak dikkate alınmıştır. Viskoz olmayan Burgers denkleminin çözümleri de benzer bir süreç izler. Bu tezin ana bölümü, Hopf-Cole dönüşümü kullanılarak yapılan çözümlere ayrılmıştır. Sonraki bölümde model nümerik methodlar uygulanarak incelenmektedir. Nümerik sonuçlar için BTCS ve FTCS kullandık. Analitik ve nümerik sonuçların karşılaştırılması ve hata analizi, bu çalışmanın son bölümünde birkaç örnekle açıklanmıştır.

Anahtar Kelimeler: Viskoz Burgers denklemi, Hopf-Cole dönüşümü, Değişkenlerine ayırma metodu, Sonlu Farklar Metodu

To my grandfather Hamit

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I love you all so much, it would not be possible without your support and trust. Stay in love.

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TABLE OF CONTENTS

LIST OF TABLES

TABLES

LIST OF FIGURES

FIGURES

LIST OF ABBREVIATIONS

BTCS	Backward Time Centered Space
BVP	Boundary Value Problem
FDM	Finite Difference Method
FEM	Finite Element Method
FVM	Finite Volume Method
FTCS	Forward Time Centered Space
IBVP	Initial-Boundary Value Problem
ODE	Ordinary Differential Equation
PDE	Partial Differential Equation

CHAPTER 1

INTRODUCTION

Partial differential equations (PDEs), which are very significant especially for mathematics and physics disciplines, are mathematical equations that comprise an unknown function dependent on two or more variables and the partial derivatives of this function. The order of a PDE is the order of the highest derivative that the equation involves. A second-order PDE is of the form

$$\alpha u_{xx} + \beta u_{xt} + \gamma u_{tt} + \delta u_x + \epsilon u_t + \zeta u = \eta \tag{1.0.1}$$

where $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$ and η are coefficients and at least one of the coefficients $\alpha, \beta,$ and γ is nonzero. There are three different classifications for second-order quasilinear PDEs. The Equation (1.0.1) is

- elliptic PDE if the discriminant $\Delta = \beta^2 - 4\,\alpha\gamma < 0$,
- hyperbolic PDE if the discriminant $\Delta = \beta^2 - 4\,\alpha\gamma > 0$,
- parabolic PDE if the discriminant $\Delta = \beta^2 - 4\,\alpha\gamma = 0$.

In this study, the viscous Burgers equation in (1.0.2), which were first introduced by H. Bateman and later studied by J. Martinus Burgers [6], is analyzed. Burgers became known for developing a mathematical model of turbulence after Bateman first proposed the Equation (1.0.2). Therefore, the Equation (1.0.2) is also called the Bateman-Burgers equation. This equation is a quasilinear PDE and its general form with initial condition and boundary conditions is described as

$$u_x + uu_t = vu_{xx},\ 0 < x < L,\ t > 0, \tag{1.0.2}$$

$$u(x, 0) = \phi(x), \tag{1.0.3}$$

$$u(0, t) = \psi_1(t),\ u(L, t) = \psi_2(t), \tag{1.0.4}$$

where vu_{xx} is called as viscosity term and $\phi(x), \psi_1(x)$ and $\psi_2(x)$ are arbitrary functions. If the right hand side of the Equation (1.0.2) is equal to zero, that is $v = 0$, this equation is called the inviscid Burgers equation. If v is a positive constant number in (1.0.2), the equation is called the viscous (or viscid) Burgers equation.

In the years that followed, Hopf and Cole [6] developed the transformation

$$u(x, t) = -2v\frac{\theta_x}{\theta} \tag{1.0.5}$$

that converts the viscous Burgers equation into the heat equation. The following Equation (1.0.6) is the heat equation obtained by applying this transformation to the Burgers equation:

$$\theta_x - v\theta_{xx} = 0,\ 0 < x < L,\ t > 0, \tag{1.0.6}$$

$$\theta(x, 0) = \bar{\phi}(x), \tag{1.0.7}$$

$$\theta_x(0, t) = \bar{\psi}_1(t),\ \theta_x(0, t) = \bar{\psi}_2(t). \tag{1.0.8}$$

where $\bar{\phi}(x), \bar{\psi}_1(t)$ and $\bar{\psi}_2(t)$ denote functions obtained after the Hopf-Cole transformation to functions $\phi(x), \psi_1(x)$ and $\psi_2(x)$. In this study, firstly, the classical solution of the viscous Burgers equation is investigated by considering the initial-boundary value problem (IBVP) given above is carried out by means of the method of separation of variables and the Fourier transform. Then this problem is solved numerically using explicit and implicit numerical methods and compared with the classical solution. In particular, Backward Time Centered Space (BTCS) method and Forward Time Centered Space (FTCS) method are used as implicit and explicit methods, respectively.

Burgers equation has been solved using similar or distinct methods in the literature. Numerical solutions obtained by applying Hopf-Cole transform and Crank-Nicolson method to the Burgers equation for different viscosity coefficients and classical solutions are compared in [11]. The multisymplectic box method, which is a fully implicit method, has been applied to Burgers equation and it has been observed that this method gives more precise results compared to the explicit and semi-explicit methods [19]. The one-dimensional heat problem is solved using finite difference method (FDM) and finite element method (FEM), and the numerical solutions obtained as a result of the use of these two methods were graphically compared with the classical solution [13]. The implicit logarithmic finite difference approach was used to solve a system of equations made up of two one-dimensional Burgers equations to get numerical results, which were then compared to the classical solutions [17]. In [9], numerical solution of the viscous Burgers equation derived using the explicit logarithmic finite difference method and classical solutions were compared.

CHAPTER 2

BURGERS EQUATION

In this chapter, we start by considering the conservation law in one-dimensional case, which is a first-order quasilinear partial differential equation, after giving essential background information on quasilinear partial differential equations. Then the solutions of the linear advection equation and the inviscid Burgers equation are examined. The main part of this chapter is devoted to the viscous Burgers equation.

2.1 First-Order Partial Differential Equations

A first-order partial differential equation is a mathematical equation that contains the unknown function u of two independent variables t and x, and its first-order partial derivatives with respect to the independent variables. The general form of the first-order partial differential equations is

$$F(t, x, u, u_t, u_x) = 0. \tag{2.1.1}$$

In this study, we focus on quasilinear partial differential equations, a particular form of PDE. General form of the first-order quasilinear PDE is given by

$$\alpha(t, x, u)u_t + \beta(t, x, u)u_x = \gamma(t, x, u), \tag{2.1.2}$$

where the coefficients α, β, γ are non-zero and continuously differentiable functions. This solution can also be stated implicitly as $F(t, x, u) = u(t, x) = u$, where $u = u(t, x)$ is a surface in $\mathbb{R}^3$. The normal vector to the surface F is $(u_t, u_x, -1)$.

When the Equation (2.1.2) is written as an inner product of the following

$$(\alpha, \beta, \gamma) \cdot (u_t, u_x, -1) = 0, \tag{2.1.3}$$

one can easily observe that Equations (2.1.2) and (2.1.3) are equivalent. As a result, (α, β, γ) is perpendicular to the normal vector to surface F, and the vector field (α, β, γ) must lie in the tangent plane to F.

The Equation (2.1.2) can be solved by characteristic curves (or simply characteristics) (2.1.4) given by

$$\frac{dt}{\alpha(t,x,u)} = \frac{dx}{\beta(t,x,u)} = \frac{du}{\gamma(t,x,u)}. \tag{2.1.4}$$

In this way, the partial differential equation (2.1.2) transforms into an ordinary differential equation and this method is called characteristics method.

Let $\varphi(t,x,u)$ and $\psi(t,x,u)$ be the solution curves obtained from the solution of (2.1.4), then the general solution to the first-order quasilinear PDE (2.1.2) can be written as

$$f(\varphi, \psi) = 0,$$

where f is an arbitrary function of two variables φ and ψ.

2.2 Conservation Laws

A conservation law, a fundamental physics principle,which do not change some physical properties such as energy, mass, momentum, etc., over time. In one-dimensional space a conservation law, which is a first–order homogeneous PDE, can be written in differential form as

$$u_t + (f(u))_x = 0, \tag{2.2.1}$$

where $u = u(x,t)$ and t are the unknown function and time, respectively and $f : \mathbb{R} \to$

$\mathbb{R}$ represents the flux function. Let the coefficients be $\alpha(t, x, u) = 1$, $\beta(t, x, u) = f^{'}(u)$, and $\gamma(t, x, u) = 0$ in (2.1.2). In that case, we can observe that Equation (2.2.1), which is a special case of (2.1.2), is also a quasilinear PDE. That is, Equation (2.2.1) can be written in the quasilinear form

$$u_t + f^{'}(u)u_x = 0. \tag{2.2.2}$$

If we integrate Equation (2.2.1) with respect to x from a to b, then we obtain

$$\begin{aligned}
\frac{d}{dt}\int_a^b u(x,t)dx &= \int_a^b u_t(x,t)dx \\
&= -\int_a^b f(u(x,t))_x dx \\
&= f(u(a,t)) - f(u(b,t)).
\end{aligned} \tag{2.2.3}$$

Equation (2.2.3) is the integral form of the conservation law. Here $u(x, t)$ is the *conserved quantity* since u is neither generated nor destroyed.

2.3 Linear Advection Equation

One of the most fundamental equations in mathematics is one dimensional linear advection equation, also known as the transport equation. The general form of the equation is

$$u_t + cu_x = 0, \tag{2.3.1}$$

where $u = u(x, t)$, $x \in \mathbb{R}$. It is advected by a nonzero constant c at time t. The direction of wave propagation is explained by the sign of c. When c is positive, the wave propagates along the x-axis in a positive direction. When c is negative, on the other hand, the wave propagates in the opposite direction of the x-axis. The wave's propagation speed is dependent on the magnitude of the constant c.

Linear advection equation can be solved by characteristics method as it is a quasilinear PDE. Supposing that $u(x,0) = u_0(x)$ is an initial condition for Equation (2.3.1), it follows that

$$\frac{dt}{1} = \frac{dx}{c} = \frac{du}{0}. \tag{2.3.2}$$

The characteristic curves for Equation (2.3.1) are described by (2.3.2). When the equalities

$$\frac{dt}{1} = \frac{du}{0} \qquad \text{and} \qquad \frac{dt}{1} = \frac{dx}{c}$$

are solved, we obtain $u = c_1$ and $x - ct = c_2$, where c_1, and c_2 are arbitrary constant. We can write the general solution of this equation in implicit form $c_1 = F(c_2)$, that is $u = F(x - ct)$. By using the initial condition, we obtain the general solution to the linear advection equation (2.3.1) as $u(x,t) = u_0(x - ct)$.

2.4 Burgers Equation

Burgers equation, which was firstly introduced by Harry Bateman [6], is also known as Bateman-Burgers equation. It is a quasilinear PDE encountered in many fields of applied mathematics. Depending on the source term, the equation may be inviscid Burgers equation or viscous Burgers equation.

2.4.1 Inviscid Burgers Equation

The general form of the inviscid Burgers equation is

$$u_t + uu_x = 0. \tag{2.4.1}$$

The Equation (2.4.1) can alternatively be stated as a conservation law

$$u_t + [f(u)]_x = 0, \text{ with } f(u) = \frac{u^2}{2}. \tag{2.4.2}$$

Although these two equations are mathematically equivalent, their numerical implementation may differ.

When we consider the similarity between the inviscid Burgers equation and linear advection equation, it follows that the solution of Equation (2.4.1) is $u(x,t) = u_0(x - ut)$. In fact, since

$$\frac{dx}{dt} = u, \qquad \frac{du}{dt} = 0,$$

by integrating these two equations, we obtain

$$x = ut + c_1, \qquad u = c_2.$$

We can write characteristic curves

$$x(t) = u_0(x_0)t + x_0$$

where $u(x,0) = u_0(x)$, and x_0 is the x-intercept of the curve. One can easily notice that characteristic curves are straight lines and they intersect with one another depending on given initial conditions. Even with smooth initial data, the solution $u(x,t)$ to the inviscid Burgers equation might become discontinuous in a finite time. When the characteristics intersect, wave breaking occurs. The time value t at which this occurs for the first time is called breaking time. For further detail, we refer to [14].

2.4.1.1 Weak Solutions

The solution of a partial differential equation can be discontinuous even if the initial value is continuous as previously stated. This discontinuity is known as shock waves. Discontinuous solutions to a differential equation are validated by expanding the class of solutions to include them. Now, assume that $\mu : \mathbb{R} \to [0, \infty)$ is a test function on a compact set. Let u be a smooth solution to the following PDE

$$u_t + (f(u))_x = 0.$$

Multiplying this equation by a test function μ gives

$$\int_0^\infty \int_{-\infty}^\infty [u_t(t,x)\mu(t,x) + (f(u(t,x)))_x\mu(t,x)]dxdt = 0$$

and then by using integration by parts it follows that

$$\begin{aligned} 0 &= \int_{-\infty}^\infty \mu u(t,x)|_0^\infty dx + \int_0^\infty \mu f(u)|_{t=-\infty}^{t=\infty} dt - \int_0^\infty \int_{-\infty}^\infty (\mu_t u(t,x) + \mu_x f(u))dxdt \\ &= -\int_0^\infty \int_{-\infty}^\infty (\mu_t u(t,x) + \mu_x f(u))dxdt - \int_{-\infty}^\infty \mu u(t,x)|_0^\infty dx. \end{aligned}$$

Applying the initial condition $u(x, 0) = u_0(x)$ gives

$$\int_0^\infty \int_{-\infty}^\infty \mu_t u(t,x) + \mu_x f(u))dxdt + \int_{-\infty}^\infty \mu u(t,x)|_0^\infty dx = 0.$$

The solution $u(x, 0)$ that satisfies this equation is called a weak solution. Many weak solutions may exist for a given initial value problem for a hyperbolic PDE. We need some shock conditions linking the jumps of u across the discontinuity so as to avoid this non-uniqueness. In this case, Rankine-Hugoniot jump relation and entropy conditions are used. Further details we address the following reference [14].

2.4.2 Viscous Burgers Equation

The general form of the Burgers equation is of the form

$$u_t + uu_x = vu_{xx}, \tag{2.4.3}$$

where v is a positive constant number. In order to solve this equation, initial and boundary conditions are defined. Such a problem is called an initial boundary value problem (IBVP). It is possible to solve this problem in different ways. In this study, the method of separation of variables and the Fourier transform are used to solve the IBVP.

2.4.2.1 Method of Separation of Variables

The method of separation of variables is one of the most commonly used techniques to solve a PDE. On the other hand, not all PDEs can be solved using the method of separation variables. The PDE and its boundary conditions must be linear and homogeneous in order to apply the method of separation of variables [10]. The following definitions provide an explanation for the linear homogeneous PDE.

For instance, the viscous Burgers equation

$$u_t + uu_x = u_{xx} \tag{2.4.4}$$

is not a linear equation due to the fact that the dependent variable u and its derivative u_x are multiplied. On the other hand, the obtained heat equation when the Hopf-Cole transform is implemented to (2.4.4)

$$\theta_t - \theta_{xx} = 0 \tag{2.4.5}$$

is a linear equation.

In this context, the method can be applied to Equation (2.4.5) which is obtained after the Hopf-Cole transformation applied to Equation (2.4.4). How the method is applied is described below.

The objective of the separation of variables technique is to obtain a product solution in the form

$$\theta(x,t) = \phi(x)\, G(t), \tag{2.4.6}$$

where ϕ is a function of x only and G is a function of t only. In other words, the solution of the given PDE, $\theta(x,t)$, is the product of two functions that depend only on x and only on t. Hence we have

$$(\phi(x)\, G(t))_t = v\,(\phi(x)\, G(t))_{xx}.$$

Yielding

$$\phi(x)\, G_t(t) = v\, G(t)\, \phi_{xx}(x).$$

The above equation is written up as follows

$$\frac{1}{v\, G(t)}\, G_t(t) = \frac{1}{\phi(x)}\, \phi_{xx}(x). \tag{2.4.7}$$

From this equation, it is clear that the left hand side of the equation depends only on the variable t and the right hand side depends on only on the variable x. That is, in this case the equation is separated into variables. Only when the terms on the right and left hand sides of the equation are equal to the same constant the equation can be satisfied, i.e.

$$\frac{1}{v\, G(t)}\, G_t(t) = \frac{1}{\phi(x)}\, \phi_{xx}(x) = -\lambda, \tag{2.4.8}$$

where λ is an arbitrary constant.

In this way, two distinct ODEs are derived from equation (2.4.8):

$$G_t(t) = -v\,\lambda G(t), \tag{2.4.9}$$

$$\phi_{xx}(x) = -\lambda\phi(x). \tag{2.4.10}$$

As a result, two ordinary differential equations that are simple to solve are generated by the method of separation of variables.

We next examine at some examples of how the viscous Burgers equation is solved.

Consider the initial value problem given below

$$\begin{cases} u_t + uu_x = vu_{xx}, \qquad t > 0, \qquad v > 0\,, \\ u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \end{cases} \tag{2.4.11}$$

This problem can be solved by using the Hopf-Cole transform, which is a mathematical transformation defined as follows

$$u(x,t) = -2v\frac{\theta_x}{\theta}\,. \tag{2.4.12}$$

We apply this transformation to the viscous Burgers equation and its initial condition. We have

$$\begin{aligned} u_t &= -2v\left(\frac{\theta\theta_{xt} - \theta_x\theta_t}{\theta^2}\right) = 2v\left(\frac{\theta_x\theta_t - \theta\theta_{xt}}{\theta^2}\right), \\ u_x &= -2v\left(\frac{\theta_{xx}\theta - \theta_x^2}{\theta^2}\right), \\ u_{xx} &= -2v\left(\frac{[(\theta_x\theta_{xx} + \theta\theta_{xxx}) - (2\theta_x\theta_{xx})]\theta^2 - 2\theta\theta_x(\theta\theta_{xx} - \theta_x^2)}{\theta^4}\right) \\ &= -2v\left(\frac{\theta\theta_{xxx} - \theta_x\theta_{xx}}{\theta^2} + \frac{2\theta\theta_x\theta_{xx} - 2\theta_x^3}{\theta^4}\right). \end{aligned}$$

When these are substituted in Equation (2.4.11), we obtain

$$\frac{2v(-\theta\theta_{xt}+\theta_x(\theta_t-v\theta_{xx})+v\theta\theta_{xxx})}{\theta^2}=0 \Leftrightarrow \quad -\theta\theta_{xt}+\theta_x(\theta_t-v\theta_{xx}+v\theta\theta_{xxx})=0$$

$$\Leftrightarrow \quad \theta_x(\theta_t-v\theta_{xx})=\theta(\theta_{xt}-v\theta_{xxx})$$

$$\Leftrightarrow \quad \theta_x(\theta_t-v\theta_{xx})=\theta(\theta_t-v\theta_{xx})_x$$

The last equation is satisfied if and only if both sides of the equation are equal to zero. From here it can be easily seen that

$$\theta_t - v\theta_{xx} = 0. \tag{2.4.13}$$

The Equation (2.4.13) is known as heat equation (or diffusion equation). We now apply the transformation (2.4.12) to initial condition. The Hopf-Cole transform can be written as follows

$$u(x,t) = -2v(\log\theta)_x \Leftrightarrow \frac{u(x,t)}{-2v} = (\log\theta)_x\,.$$

When we integrate both sides of the above equation over the interval $[0, x]$, we obtain

$$\log\theta = \int_0^x \frac{u(x,t)}{-2v}\,dx \;\Leftrightarrow\; \theta(x,t) = \exp\left(-\frac{1}{2v}\int_0^x u(y,t)\,dy\right).$$

At $t = 0$ we have

$$\theta(x,0) = \exp\left(-\frac{1}{2v}\int_0^x u(y,0)\,dy\right).$$

If the initial condition given as $u(x,0) = u_0(x)$ is applied to the above equation, we get

$$\theta(x,0) = \theta_0(x) = \exp\left(-\frac{1}{2v}\int_0^x u_0(y)dy\right).$$

Thus (2.4.11) is reduced to the following problem thanks to the Hopf-Cole transformation:

$$\begin{cases} \theta_t - v\theta_{xx} = 0, & t>0, \quad v>0\,, \\ \theta(0,x) = \theta_0(x), & x\in\mathbb{R}\,. \end{cases} \tag{2.4.14}$$

We now use the Fourier transform to solve the above heat equation. The general definition of the Fourier transform of a function $f(x)$ is as follows:

$$F\{f(x)\} = \hat{f}(\omega) = \int_{-\infty}^{\infty} \exp(-i\omega x) f(x) dx.$$

Here the function $\hat{f}(\omega)$ is called as a Fourier transform of function f. The inverse Fourier transform of $\hat{f}(\omega)$ is also defined by

$$F^{-1}\{\hat{f}(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\omega x) \hat{f}(\omega) d\omega.$$

By applying this transformation to (2.4.14), we get the following problem

$$\begin{aligned} F\{\theta_t\} &= F\{\theta_{xx}\} &\Leftrightarrow\quad &\hat{\theta}_t = -v\omega^2\hat{\theta}, \\ F\{\theta(x,0)\} &= F\{\theta_0(x)\} &\Leftrightarrow\quad &\hat{\theta}(\omega,0) = \hat{\theta}_0(x). \end{aligned} \tag{2.4.15}$$

The equation $\hat{\theta}_t = -v\omega^2\hat{\theta}$ is a separable ODE and with the general solution $\hat{\theta} = c\exp(-v\omega^2 t)$, where c is an arbitrary constant. By applying the initial condition to the general solution, $c = \hat{\theta}_0(\omega)$ is obtained. Thus the general solution of (2.4.15) is

$$\theta(\omega, t) = \hat{\theta}_0(\omega) \exp(-v\omega^2 t).$$

By using the inverse Fourier transformation F^{-1}, we obtain $\theta(x,t)$ as

$$\begin{aligned} \theta(x,t) = F^{-1}(\hat{\theta}(\omega,t)) = F^{-1}(\hat{\theta}_0(\omega)\ \exp(-v\omega^2 t)) &= \hat{\theta}_0(\omega)\ *\ F^{-1}(\exp(-v\omega^2 t)) \\ &= \theta_0(x)\ *\ F^{-1}(\exp(-v\omega^2 t)), \end{aligned}$$

where $*$ represents the convolution product.

We should find a function $\phi(x,t)$ such that $F\{\phi(x,t)\} = \exp(-v\omega^2 t)$. By means of the inverse Fourier transform formula, we get

$$\begin{aligned} \phi(x,t) = F^{-1}(\exp(-v\omega^2 t)) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-v\omega^2 t) \exp(i\omega t) d\omega \\ &= \frac{1}{\sqrt{4\pi v t}} \exp\left(-\frac{1}{4vt} x^2\right). \end{aligned}$$

Hence the general solution of (2.4.15) is

$$\theta(x,t) = \theta_0(x) \,*\, F^{-1}(\exp(-v\omega^2 t)) = \int_{-\infty}^{\infty} \phi(t, x-\omega)\theta_0(\omega)d\omega$$
$$= \frac{1}{\sqrt{4\pi v t}} \int_{-\infty}^{\infty} \theta_0(\omega) \exp\left(-\frac{(x-\omega)^2}{4vt}\right)d\omega.$$

Consequently, by substituting $\theta(x,t)$ in the Hopf-Cole transformation, we obtain the general solution of (2.4.11) as

$$u(x,t) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-\omega}{t}\theta_0(\omega) \exp\left(-\frac{(x-\omega)^2}{4vt}\right)d\omega}{\displaystyle\int_{-\infty}^{\infty} \theta_0(\omega) \exp\left(-\frac{(x-\omega)^2}{4vt}\right)d\omega}.$$

Example 2.4.1. *Consider* (2.4.3) *with the initial condition*

$$u(x,0) = \sin(\pi x), \qquad 0 < x < 1$$

and homogeneous boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad t > 0.$$

The Hopf-Cole transform can be used to solve the given IBVP. We have previously observed that the Equation (2.4.3) *is reduced to the heat equation after this transformation. Similarly, when the transformation is applied to the initial data and boundary conditions, the following results are obtained:*

$$\theta(x,0) = \exp\{-(2v)^{-1}\int_0^x u(y,0)\,dy\} = \exp\{-(2v)^{-1}\int_0^x \sin(\pi y)\,dy\}$$
$$= \exp\{(2\pi v)^{-1}[\cos(\pi x) - 1]\},$$
$$\theta_x(0,t) = \theta_x(1,t) = 0\,.$$

After applying the Hopf-Cole transformation, the problem becomes

$$\begin{cases} \theta_t - v\theta_{xx} = 0, \qquad 0 < x < 1, \\ \theta(x,0) = \exp\{(2\pi v)^{-1}[\cos(\pi x) - 1]\}, \\ \theta_x(0,t) = 0,\ \theta_x(1,t) = 0. \end{cases} \tag{2.4.16}$$

It can be observed that in (2.4.16) *the Neumann boundary conditions are obtained after applying the Hopf-Cole transform to the Dirichlet boundary conditions in the IBVP given by* ***Example 2.4.1***.

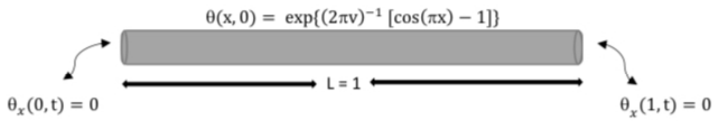

Figure 2.1: A heat bar model for the (2.4.16) with homogenous boundary conditions.

Let us solve the IBVP (2.4.16) *by via the separation of variables method by setting* $\theta(t,x) = \phi(x)G(t)$. *We solve the following two ordinary differential equations:*

$$\frac{dG}{dt} = -v\lambda G, \qquad \frac{d^2\phi}{dx^2} = -\lambda\phi.$$

Now, let us plug $\theta(x,t) = \phi(x)G(t)$ *into the boundary conditions,*

$$G(t)\frac{d\phi}{dx}(0) = 0, \qquad G(t)\frac{d\phi}{dx}(1) = 0\,.$$

G(t) must be non-zero so that the solution is not a trivial solution. Therefore, we have

$$\frac{d\phi}{dx}(0) = 0, \qquad \frac{d\phi}{dx}(1) = 0\,.$$

We solve this boundary value problem. There are three cases with respect to λ*:*

- *If* $\lambda > 0$, *then the general solution to the differential equation is* $\phi(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$. *Applying the boundary conditions, it follows that*

$$\frac{d\phi}{dx} = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + c_2\sqrt{\lambda}cos(\sqrt{\lambda}x),$$
$$0 = \frac{d\phi}{dx}(0) = \sqrt{\lambda}c_2 \qquad \Rightarrow \qquad c_2 = 0,$$
$$0 = \frac{d\phi}{dx}(1) = -\sqrt{\lambda}c_1\sin(\sqrt{\lambda}).$$

We start with assuming the condition $\lambda > 0$ *so* $c_1\sin(\sqrt{\lambda})$ *must be equal to zero. On the other hand, if* $c_1 = 0$*, then we obtain trivial solution. To get nontrivial solution, that is,* $c_1 \neq 0$, *and* $\sin(\sqrt{\lambda}) = 0$ *we have*

$$\sin(\sqrt{\lambda}) = 0 \qquad \Rightarrow \qquad \sqrt{\lambda} = n\pi, \qquad n = 1, 2, ...$$

For this boundary value problem eigenvalues and their corresponding eigenfunctions are

$$\lambda_n = n^2\pi^2, \qquad \phi_n(x) = \cos(n\pi x), \qquad n = 1, 2, \dots .$$

- *If* $\lambda = 0$, *then the general solution is* $\phi(x) = c_1 + c_2x$. *We apply the boundary conditions*

$$\frac{d\phi}{dx} = c_2,$$
$$0 = \frac{d\phi}{dx}(0) = c_2,$$

and we get $\phi(x) = c_1$ *since* $c_2 = 0$. *Hence for* $\lambda = 0$, *the eigenfunction corresponding to this eigenvalue is* $\phi(x) = 1$.

- *If* $\lambda < 0$, *then the general solution to the equation is* $\phi(x) = c_1\cosh(\sqrt{-\lambda}x) + c_2\sinh(\sqrt{-\lambda}x)$. *We obtain the followings by applying boundary conditions:*

$$\frac{d\phi}{dx} = c_1\sqrt{-\lambda}\sinh(\sqrt{-\lambda}x) + c_2\sqrt{-\lambda}\cosh(\sqrt{-\lambda}x),$$
$$0 = \frac{d\phi}{dx}(0) = \sqrt{-\lambda}c_2 \qquad \Rightarrow \qquad c_2 = 0, \qquad \textit{and}$$
$$0 = \frac{d\phi}{dx}(1) = \sqrt{-\lambda}c_1\sinh(\sqrt{-\lambda}).$$

It can be observed that $\sinh(\sqrt{-\lambda}) \neq 0$ *due to* $\sqrt{-\lambda} \neq 0$. *In that case,* c_1 *must be equal to zero. As a result, we get* $\phi(x) = 0$.

To sum up for this problem eigenvalues and eigenfunctions are

$$\lambda_n = n^2\pi^2, \qquad \phi_n(x) = \cos(n\pi x), \qquad n = 1, 2, \dots .$$
$$\lambda_0 = 0, \qquad \phi_0(x) = 1.$$

It can be written in the following form

$$\lambda_n = n^2\pi^2, \qquad \phi_n(x) = \cos(n\pi x), \qquad n = 0, 1, 2, \dots .$$

$\dfrac{dG}{dt} = -v\lambda G$ *is a linear first-order separable differential equation with general solution*

$$G(t) = c\exp(-vn^2\pi^2 t),$$

where c *is an arbitrary constant. Hence the general solution of Equation* (2.4.16) *is*

$$\theta(x,t) = \sum_{n=0}^{\infty} A_n \cos(n\pi x)\exp(-vn^2\pi^2 t)\,.$$

When θ *is substituted in the Hopf-Cole transform equation* (2.4.12)*, the general solution to the given BVP is*

$$u(x,t) = 2\pi v \frac{\displaystyle\sum_{n=1}^{\infty} nA_n \sin(n\pi x)\exp(-vn^2\pi^2 t)}{\displaystyle A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)\exp(-vn^2\pi^2 t)}$$

with the Fourier coefficients

$$A_n = \begin{cases} \displaystyle\int_0^1 \exp\{(2\pi v)^{-1}[\cos(\pi x) - 1]\}dx & n = 0\,, \\ \\ \displaystyle 2\int_0^1 \exp\{(2\pi v)^{-1}[\cos(\pi x) - 1]\}\cos(n\pi x)dx & n \neq 0\,. \end{cases}$$

Example 2.4.2. *Consider* (2.4.3) *with the initial condition*

$$u(x,0) = 2x(1-x), \qquad 0 < x < 1$$

and the homogeneous boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad t > 0.$$

Here take $v = 1$*. Applying the Hopf-Cole transformation we utilized to solve the previous example to the given problem. In this instance the following IBVP is obtained after applying the transform:*

$$\begin{cases} \theta_t - \theta_{xx} = 0, & 0 < x < 1, \\ \theta(x,0) = \exp\{\frac{1}{6}(2x^3 - 3x^2)\}, & \\ \theta_x(0,t) = 0, \ \theta_x(1,t) = 0. & \end{cases} \tag{2.4.17}$$

The separation of variables method can be used since the equation and boundary conditions given by (2.4.17) *are linear and homogeneous. The general solution to the problem* (2.4.17) *can be found by employing similar calculations as*

$$\theta(x,t) = \sum_{n=0}^{\infty} A_n \cos(n\pi x) \exp\left(-vn^2\pi^2 t\right). \tag{2.4.18}$$

*Solution to the IBVP given in **Example 2.4.2** can be expressed by substituting the solution* $\theta(x,t)$ *in* (2.4.12)*. This gives*

$$u(x,t) = 2\pi v \frac{\sum_{n=1}^{\infty} nA_n \sin(n\pi x) \exp\left(-vn^2\pi^2 t\right)}{A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \exp\left(-vn^2\pi^2 t\right)}$$

with the Fourier coefficients

$$A_n = \begin{cases} \int_0^1 \exp\{\frac{1}{6}(2x^3 - 3x^2)\} dx & n = 0\,, \\ \\ 2\int_0^1 \exp\{\frac{1}{6}(2x^3 - 3x^2)\} \cos(n\pi x) dx & n \neq 0\ . \end{cases}$$

In both examples we examined above, the initial conditions are continuous functions. Next we examine how to solve the IBVP whose initial condition is a piecewise function.

Example 2.4.3. *Consider* (2.4.3) *with the piecewise initial condition*

$$u(0,x) = \begin{cases} 0, & 0 < x \leq \dfrac{1}{2}, \\ -\dfrac{2}{x+1}, & \dfrac{1}{2} < x < 1, \end{cases} \tag{2.4.19}$$

and the homogeneous boundary conditions $u(0,t) = u(1,t) = 0$.

Applying the transformation (2.4.12) *yields the initial and boundary conditions listed below:*

$$\theta(0,x) = \begin{cases} 1, & 0 < x \leq \frac{1}{2}, \\ x+1, & \frac{1}{2} < x < 1, \end{cases} \tag{2.4.20}$$

and

$$\theta_x(0,t) = \theta_x(1,t) = 0. \tag{2.4.21}$$

The solution of the IBVP given by the heat equation with the method of separation of variables is

$$\theta(x,t) = \sum_{n=0}^{\infty} A_n \cos(n\pi x) \exp\left(-vn^2\pi^2 t\right) \tag{2.4.22}$$

where the Fourier coefficients are

$$A_n = \begin{cases} \displaystyle\int_0^{1/2} dx + \int_{1/2}^{1} (x+1)dx, & n = 0\,, \\ \\ \displaystyle 2\int_0^{1/2} \cos(n\pi x)dx + 2\int_{1/2}^{1} (x+1)\cos(n\pi x)dx, & n \neq 0\,. \end{cases}$$

The coefficients A_0 and A_n are found as follows:

$$n = 0 \Rightarrow A_0 = x\Big|_0^{1/2} + (\frac{x^2}{2} + x)\Big|_{1/2}^{1} = \frac{11}{8}$$

$$n \neq 0 \Rightarrow A_n = 2\frac{\sin(n\pi x)}{n\pi}\Big|_0^{1/2} + 2(x+1)\frac{\sin(n\pi x)}{n\pi}\Big|_{1/2} 1 + 2\frac{\cos(n\pi x)}{n^2\pi^2}\Big|_{1/2}^{1}$$

$$A_n = -\frac{1}{n\pi}\sin(\frac{n\pi}{2}) + \frac{2}{n^2\pi^2}[(-1)^n - \cos(\frac{n\pi}{2})], \qquad n = 1, 2, ...$$

Hence the general solution of the IBVP in **Example 2.4.3** is

$$u(x,t) = 2\pi \frac{\sum_{n=1}^{\infty} nA_n \sin(n\pi x) \exp\left(-n^2\pi^2 t\right)}{A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \exp\left(-n^2\pi^2 t\right)}. \qquad (2.4.23)$$

Substituting the coefficients A_0 and A_n into Equation (2.4.23), we get

$$u(x,t) = 2\pi \frac{\sum_{n=1}^{\infty}\left(-\frac{1}{\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{n\pi^2}\left[(-1)^n - \cos\left(\frac{n\pi}{2}\right)\right]\right)\sin(n\pi x)\exp(-n^2\pi^2 t))}{\frac{11}{8} + \sum_{n=1}^{\infty}\left(-\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{n^2\pi^2}\left[(-1)^n - \cos(\frac{n\pi}{2})\right]\right)\cos(n\pi x)\exp(-n^2\pi^2 t)}.$$

CHAPTER 3

IMPLEMENTATION OF NUMERICAL METHODS

In the analysis of engineering problems and in revealing the underlying physical mechanisms of these problems, PDEs play an important role. Directly calculating and analyzing modern engineering problems, which we come across rather frequently today and which are described through PDEs, are quite time-consuming and complicated. For this reason, numerical methods are utilized to calculate the solution of PDEs. As package programs and subprograms have developed through time, numerical methods have begun to be preferred over classical methods due to their ease of use and inexpensive cost, even for problems that can be solved classically. So, briefly, a numerical method can be described as a computer-based approach to tackling a mathematical problem that usually has no classical solution. Since the 18th century, as computer capacities and computing capabilities have advanced, the variety of numerical methods implemented to solve ODEs and PDEs has grown considerably.

To solve PDEs, it can be utilized a variety of numerical methods. The finite element method (FEM), the finite volume method (FVM), and the finite difference method (FDM) are the most frequently used numerical methods. The FDM implemented to solve the viscous Burgers equation is the subject of this thesis. The objective of the finite difference method, first used by Euler, is to solve the differential equation by approximating the derivatives with finite differences. This method converts ODEs and PDEs into systems of linear equations, even if they are not linear. Due to the efficiency with which modern computers can carry out these algebraic operations and the simplicity of their implementation, FDM is frequently used in numerical analysis.

3.0.1 Finite Difference Approximations

The FDM is based on a function's Taylor series expansion. Consider a continuous function θ. Using the Taylor series expansion, the value of this function at the neighboring point $x = x_0$ can be written as

$$\begin{aligned} \theta(x_0 + \Delta x) &= \theta(x_0) + \sum_{n=1}^{\infty} \frac{(\Delta x)^n}{n!}\theta^{(n)}(x_0) \\ &= \theta(x_0) + \Delta x\,\theta'(x_0) + ... + \frac{(\Delta x)^n}{n!}\theta^{(n)}(x_0) + R_n, \end{aligned} \tag{3.0.1}$$

where R_n is called remainder term (or truncation error) and defined as

$$R_n = \frac{(\Delta x)^{n+1}}{(n+1)!}\theta^{(n+1)}(\zeta_{n+1}), \qquad x_0 < \zeta_{n+1} < x. \tag{3.0.2}$$

Particularly for $n = 1$, one gets

$$\theta'(x_0) = \frac{\theta(x_0 + \Delta x) - \theta(x_0)}{\Delta x} - \underbrace{\frac{\Delta x}{2}\,\theta''(\zeta_2)}_{\mathcal{O}(\Delta x)}. \tag{3.0.3}$$

Since the term denoted by $\mathcal{O}(\Delta x)$ is less than Δx, it is a negligible error term. So, the forward difference approximation is obtained as follows:

$$\theta'(x_0) \approx \frac{\theta(x_0 + \Delta x) - \theta(x_0)}{\Delta x}. \tag{3.0.4}$$

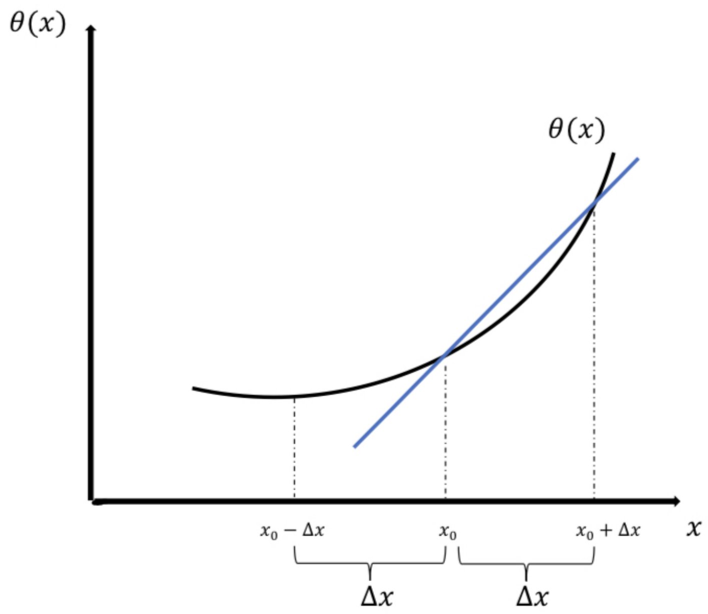

Figure 3.1: Forward Difference Approximation

In the forward difference approximation given with Figure 3.4, the black curve is the true slope while the blue curve is the approximated slope.

Similarly,

$$\begin{aligned}\theta(x_0 - \Delta x) &= \theta(x_0) + \sum_{n=1}^{\infty}(-1)^n \frac{(\Delta x)^{(n)}}{n!}\theta^{(n)}(x_0) \\ &= \theta(x_0) - \Delta x\, \theta'(x_0) + \ldots + \frac{(\Delta x)^n}{n!}\theta^{(n)}(x_0) + R_n,\end{aligned} \tag{3.0.5}$$

where R_n is remainder term. For $n = 1$,

$$\theta'(x_0) = \frac{\theta(x_0 - \Delta x) - \theta(x_0)}{-\Delta x} + \underbrace{\frac{\Delta x}{2}\,\theta''(\xi_2)}_{\mathcal{O}(\Delta x)}. \tag{3.0.6}$$

Then from (3.0.6) the backward difference approximation

$$\theta'(x_0) \approx \frac{\theta(x_0 - \Delta x) - \theta(x_0)}{-\Delta x} = \frac{\theta(x_0) - \theta(x_0 - \Delta x)}{\Delta x} \tag{3.0.7}$$

is obtained. In the backward difference approximation given by the Figure 3.2, the black curve is the true slope while the yellow curve is the approximated slope.

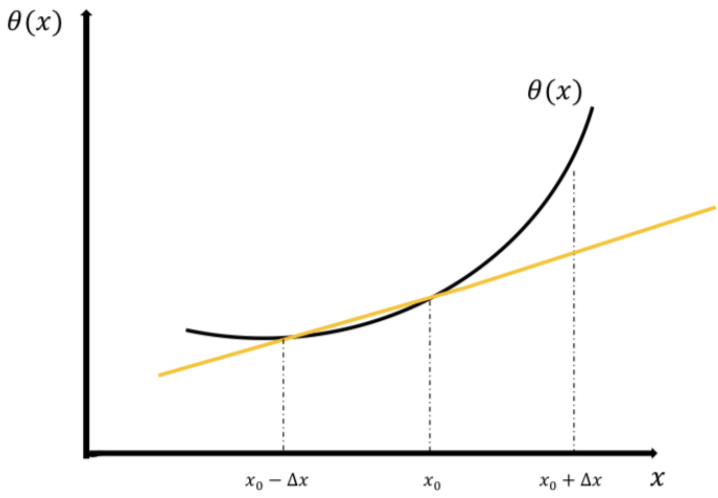

Figure 3.2: Backward Difference Approximation

Substracting (3.0.1) from (3.0.5) lead to

$$\theta(x_0 + \Delta x) - \theta(x_0 - \Delta x) = 2\,(\Delta x)\theta'(x_0) + \frac{(\Delta x)^3}{3}\theta'''(\zeta_3).$$

By arranging the above equation one obtains

$$\theta'(x_0) = \frac{\theta(x_0 + \Delta x) - \theta(x_0 - \Delta x))}{2\Delta x} - \underbrace{\frac{(\Delta x^2)}{6}\theta'''(\zeta_3)}_{\mathcal{O}((\Delta x)^2)}. \tag{3.0.8}$$

Hence the centered difference approximation gives

$$\theta'(x_0) \approx \frac{\theta(x_0 + \Delta x) - \theta(x_0 - \Delta x)}{2\Delta x} \tag{3.0.9}$$

more precise approximation. In the centered difference approximation given by the Figure 3.3, the black curve is the true slope while the green curve is the approximated slope.

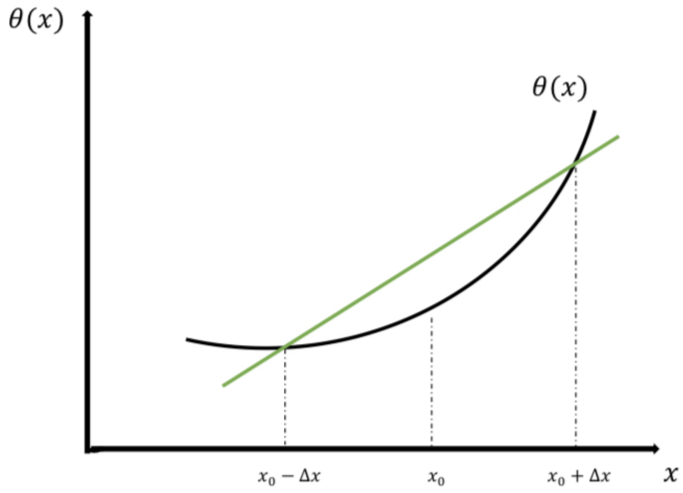

Figure 3.3: Centered Difference Approximation

3.0.1.1 The Heat Equation's FTCS Approximation

Consider the following IBVP

$$\begin{cases} \theta_t - v\theta_{xx} = 0, \qquad 0 < x < L, \\ \theta(x, 0) = f(x) \\ \theta_x(0, t) = \alpha(t), \ \theta_x(L, t) = \beta(t). \end{cases} \tag{3.0.10}$$

Let us first discretize space and time interval, respectively,

$$h = \Delta x = \frac{L}{m}, \qquad x_i = \alpha(t) + i\Delta x, \qquad i = 0, 1, ..., m$$
$$k = \Delta t = \frac{t_f}{n}, \qquad t_j = j\Delta t, \qquad j = 0, 1, ..., n$$

where t_f is a final time. At each mesh point (x_i, t_j) the value of θ is denoted by

$$\theta(x_i, t_j) = \theta(ih, jk) = \theta_{i,j}. \tag{3.0.11}$$

The forward difference approximation for the time derivative θ_t is

$$\theta_t(x,t) \approx \frac{\theta(x,t+\Delta t) - \theta(x,t)}{\Delta t}, \tag{3.0.12}$$

and the central difference approximation for the second-order space derivative θ_{xx} is

$$\theta_{xx}(x,t) \approx \frac{\theta(x+\Delta x,t) - 2\theta(x,t) + \theta(x-\Delta x,t)}{\Delta x^2}. \tag{3.0.13}$$

When (3.0.12) and (3.0.13) are substituted in the equation given in (3.0.10), it is obtained

$$\frac{\theta(x,t+\Delta t) - \theta(x,t)}{\Delta t} - v\frac{\theta(x+\Delta x,t) - 2\theta(x,t) + \theta(x-\Delta x,t)}{\Delta x^2} = 0. \tag{3.0.14}$$

In the approximation (3.0.14), if it is taken $x = x_i$, $t = t_j$, $\Delta x = h$, and $\Delta t = k$ we get

$$\frac{\theta(x_i,t_j+k) - \theta(x_i,t_j)}{k} = v\frac{\theta(x_i-h,t_j) - 2\theta(x_i,t_j) + \theta(x_i+h,t_j)}{h^2}. \tag{3.0.15}$$

By using the equalities $x_i = ih$ and $t_j = jk$, the Equation (3.0.15) can be written in the following form:

$$\frac{\theta(ih,jk+k) - \theta(ih,jk)}{k} = v\frac{\theta(ih-h,jk) - 2\theta(ih,jk) + \theta(ih+h,jk)}{h^2},$$

$$\frac{\theta(ih,(j+1)k) - \theta(ih,jk)}{k} = v\frac{\theta((i-1)h,jk) - 2\theta(ih,jk) + \theta((i+1)h,jk)}{h^2}.$$

From the Equation (3.0.11), it follows that

$$\theta_{i,j+1} - \theta_{i,j} = \frac{kv}{h^2}(\theta_{i-1,j} - 2\theta_{i,j} + \theta_{i+1,j}),$$

where $r = \frac{vk}{h^2} = \frac{v\Delta t}{\Delta x^2}$. Rearranging the above equation yields the FTCS scheme:

$$\theta_{i,j+1} = r\theta_{i-1,j} + (1-2r)\theta_{i,j} + r\theta_{i+1,j}. \tag{3.0.16}$$

In Figure 3.4, the points $\theta_{i-1,j}$, $\theta_{i,j}$ and $\theta_{i+1,j}$ are known and the point $\theta_{i,j+1}$ is calculated using these points.

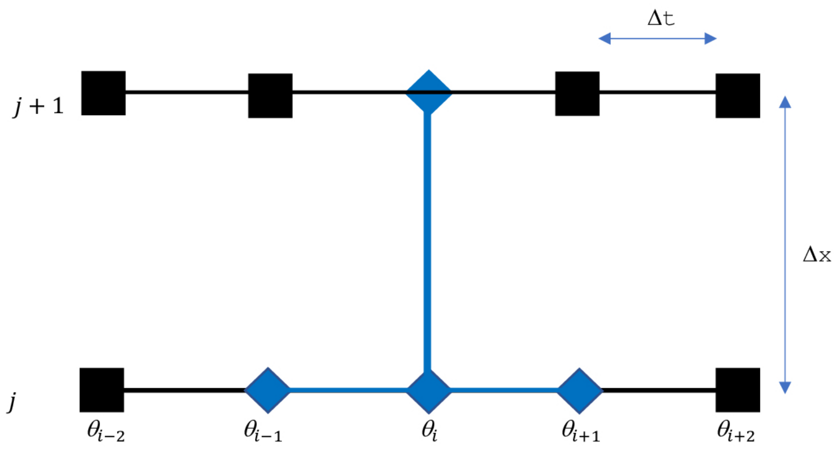

Figure 3.4: The FTCS scheme

3.0.1.2 The Heat Equation's BTCS Approximation

In this method, the space derivative is central and the backward difference is applied to the time derivative at (x_i, t_j). Consider the IBVP (3.0.10). The heat equation is discretized by using the BTCS method. The backward-difference approximation for the time derivative θ_t is

$$\theta_t(x,t) \approx \frac{\theta(x,t) - \theta(x,t-\Delta t)}{\Delta t}, \tag{3.0.17}$$

and the central difference approximation for the second-order space derivative θ_{xx} is

$$\theta_{xx}(x,t) \approx \frac{\theta(x+\Delta x,t) - 2\theta(x,t) + \theta(x-\Delta x,t)}{\Delta x^2}. \tag{3.0.18}$$

Substituting (3.0.17) and (3.0.18) in the equation given in (3.0.10) gives

$$\frac{\theta(x,t) - \theta(x,t-\Delta t)}{\Delta t} - v\frac{\theta(x+\Delta x,t) - 2\theta(x,t) + \theta(x-\Delta x,t)}{\Delta x^2} = 0. \tag{3.0.19}$$

Take $x = x_i$, $t = t_j$, $\Delta x = h$, and $\Delta t = k$ in (3.0.19), it follows that

$$\frac{\theta(x_i,t_j) - \theta(x_i,t_j-k)}{k} = v\frac{\theta(x_i-h,t_j) - 2\theta(x_i,t_j) + \theta(x_i+h,t_j)}{h^2}. \tag{3.0.20}$$

The Equation (3.0.20) can be stated in the following form by using (3.0.11):

$$\frac{\theta(ih,jk) - \theta(ih,(j-1)k)}{k} = v\frac{\theta((i-1)h,jk) - 2\theta(ih,jk) + \theta((i+1)h,jk)}{h^2}.$$

When the equation is arranged by considering the (3.0.11), we get

$$\theta_{i,j} - \theta_{i,j-1} = r\theta_{i-1,j} - 2r\theta_{i,j} + r\theta_{i+1,j}.$$

By replacing $j-1$ with j and j with $j+1$, the BTCS scheme is obtained as follows:

$$\theta_{i,j} = -r\theta_{i-1,j+1} + (2r+1)\theta_{i,j+1} - r\theta_{i+1,j+1}. \tag{3.0.21}$$

where $r = \dfrac{vk}{h^2}$. In Figure 3.5, the point $\theta_{i,j-1}$ is known and the unknown points $\theta_{i-1,j}, \theta_{i,j}$ and $\theta_{i+1,j}$ are calculated by using the LU decomposition method.

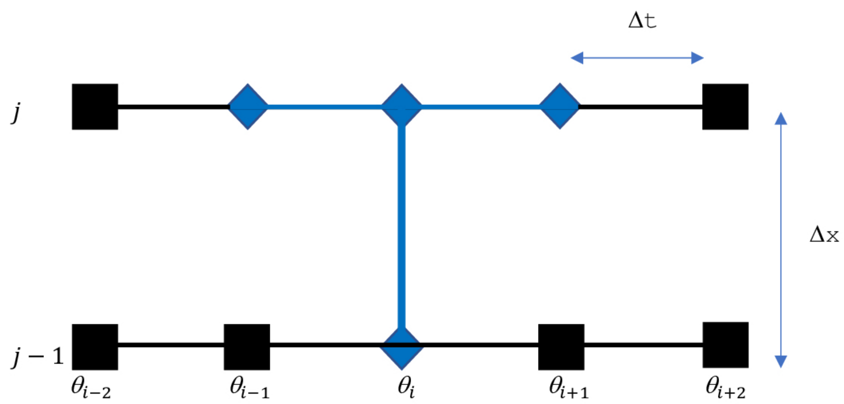

Figure 3.5: The BTCS scheme

Example 3.0.1. *Consider the following IBVP which has been classically solved in the Chapter 2*

$$\begin{cases} u_t + uu_x = vu_{xx}, & 0 < x < 1, \\ u(x,0) = \sin(\pi x), \\ u(0,t) = 0, \ u(1,t) = 0. \end{cases} \tag{3.0.22}$$

Applying the transformation, the problem becomes

$$\begin{cases} \theta_t - v\theta_{xx} = 0, & 0 < x < 1, \\ \theta(x,0) = \exp\{(2\pi v)^{-1}[\cos(\pi x) - 1]\}, \\ \theta_x(0,t) = 0, \ \theta_x(1,t) = 0. \end{cases} \tag{3.0.23}$$

We apply FTCS and BTCS methods to solve this problem (3.0.23) numerically. Take $h = 0.2$ and $k = 0.02$.

Let $v = 1$. Discretizing the x spatial domain and the t time domain is the initial step.

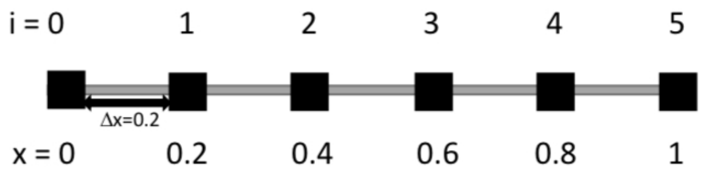

Figure 3.6: The nodes in the spatial domain for $\Delta x = 0.2$, where $x \in (0,1)$.

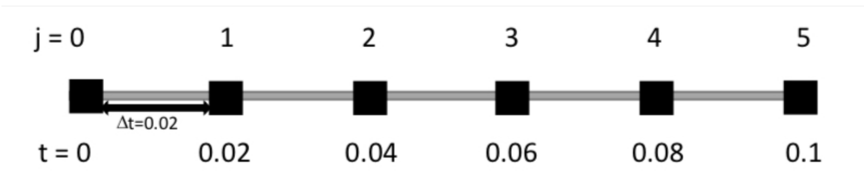

Figure 3.7: The nodes in the time domain for $\Delta t = 0.02$, where $t \in (0,0.1)$.

Using the FTCS method, the given IBVP (3.0.23) is computed. Here $r = 0.5$ for $h = 0.2$ and $k = 0.02$. Forward difference approximation is applied to the heat equation, initial condition, and Neumann boundary conditions given by the problem (3.0.23).
To obtain the FTCS approximation of the heat equation, it is sufficient to substitute r in Equation (3.0.16). In this way, the equation obtained is below.

$$\theta_{i,j+1} = 0.5\,\theta_{i-1,j} + 0.5\,\theta_{i+1,j}, \qquad i = 0,1,2,3,4, \qquad j = 0,1,2,3,4. \quad (3.0.24)$$

Applying the central difference to the boundary conditions at $x = 0$ and $x = 1$ leads to the following results, respectively.

$$0 = (\theta_x)_{0,j} = \frac{\theta_{1,j} - \theta_{-1,j}}{2\Delta x} = 0 \;\Rightarrow\; \theta_{-1,j} = \theta_{1,j},\; j = 0,1,...,5. \quad (3.0.25)$$

$$0 = (\theta_x)_{5,j} = \frac{\theta_{6,j} - \theta_{4,j}}{2\Delta x} = 0 \;\Rightarrow\; \theta_{4,j} = \theta_{6,j},\; j = 0,1,...,5. \quad (3.0.26)$$

With the initial condition $\theta_{i,0} = exp(2\pi)^{-1}[cos(\pi x_i) - 1]$, the initial values for $i = 0,1,...,5$ are calculated as follows:

$$\begin{aligned}
i = 0 &\Rightarrow \theta_{0,0} = 1,\\
i = 1 &\Rightarrow \theta_{1,0} = 0.97006,\\
i = 2 &\Rightarrow \theta_{2,0} = 0.89585,\\
i = 3 &\Rightarrow \theta_{3,0} = 0.81193,\\
i = 4 &\Rightarrow \theta_{4,0} = 0.74982,\\
i = 5 &\Rightarrow \theta_{5,0} = 0.72737.
\end{aligned}$$

Using the equations (3.0.25) and (3.0.26) obtained from the given boundary conditions and initial values computed above, we calculate the interior points. Since there are six unknowns for each j, there are six equations for each j. We have

$$
\begin{aligned}
i = 0 &\Rightarrow 0.5\,\theta_{-1,j} + 0.5\,\theta_{1,j} = \theta_{0,j+1} \\
&\Rightarrow \theta_{1,j} = \theta_{0,j+1}, \\
i = 1 &\Rightarrow 0.5\,\theta_{0,j} + 0.5\,\theta_{2,j} = \theta_{1,j+1}, \\
i = 2 &\Rightarrow 0.5\,\theta_{1,j} + 0.5\,\theta_{3,j} = \theta_{2,j+1}, \\
i = 3 &\Rightarrow 0.5\,\theta_{2,j} + 0.5\,\theta_{4,j} = \theta_{3,j+1}, \\
i = 4 &\Rightarrow 0.5\,\theta_{3,j} + 0.5\,\theta_{5,j} = \theta_{4,j+1}, \\
i = 5 &\Rightarrow 0.5\,\theta_{4,j} + 0.5\,\theta_{6,j} = \theta_{5,j+1}, \\
&\Rightarrow \theta_{4,j+1} = \theta_{5,j+1}.
\end{aligned}
\tag{3.0.27}
$$

It is clear that the equations for $i = 0$ and $i = 5$ are derived from (3.0.25) and (3.0.26). The unknown interior points are computed for $j = 0, 1, ..., 5$.

Iteration 1 : For $j = 0$ in (3.0.27).

$$
\begin{aligned}
i = 0 &\Rightarrow 0.5\,\underbrace{\theta_{-1,0}}_{\theta_{1,0}} + 0.5\,\theta_{1,0} = \theta_{0,1} \\
&\Rightarrow \theta_{1,0} = \theta_{0,1} = 0.97006, \\
i = 1 &\Rightarrow 0.5\,\underbrace{\theta_{0,0}}_{-1} + 0.5\,\underbrace{\theta_{2,0}}_{0.89585} = \theta_{1,1}\,; \quad \theta_{1,1} = 0.94792, \\
i = 2 &\Rightarrow 0.5\,\underbrace{\theta_{1,0}}_{0.97006} + 0.5\,\underbrace{\theta_{3,0}}_{0.81193} = \theta_{2,1}\,; \quad \theta_{2,1} = 0.89099, \\
i = 3 &\Rightarrow 0.5\,\underbrace{\theta_{2,0}}_{0.89585} + 0.5\,\underbrace{\theta_{4,0}}_{0.74982} = \theta_{3,1}\,; \quad \theta_{3,1} = 0.82283, \\
i = 4 &\Rightarrow 0.5\,\underbrace{\theta_{3,0}}_{0.81193} + 0.5\,\underbrace{\theta_{5,0}}_{0.72737} = \theta_{4,1}\,; \quad \theta_{4,1} = 0.76964, \\
i = 5 &\Rightarrow 0.5\,\theta_{4,0} + 0.5\,\underbrace{\theta_{6,0}}_{\theta_{4,0}} = \theta_{5,1} \\
&\Rightarrow \theta_{4,0} = \theta_{5,1} = 0.74982.
\end{aligned}
$$

Iteration 2 : For $j = 1$ in (3.0.27).

$$i = 0 \Rightarrow 0.5\,\underbrace{\theta_{-1,1}}_{\theta_{1,1}} + 0.5\,\theta_{1,1} = \theta_{0,2}$$

$$\Rightarrow \theta_{1,1} = \theta_{0,2} = 0.94792,$$

$$i = 1 \Rightarrow 0.5\,\underbrace{\theta_{0,1}}_{0.97006} + 0.5\,\underbrace{\theta_{2,1}}_{0.89099} = \theta_{1,2}\,; \quad \theta_{1,2} = 0.93052,$$

$$i = 2 \Rightarrow 0.5\,\underbrace{\theta_{1,1}}_{0.94792} + 0.5\,\underbrace{\theta_{3,1}}_{0.82283} = \theta_{2,2}\,; \quad \theta_{2,2} = 0.88537,$$

$$i = 3 \Rightarrow 0.5\,\underbrace{\theta_{2,1}}_{0.89099} + 0.5\,\underbrace{\theta_{4,1}}_{0.76964} = \theta_{3,2}\,; \quad \theta_{3,2} = 0.83031,$$

$$i = 4 \Rightarrow 0.5\,\underbrace{\theta_{3,1}}_{0.82283} + 0.5\,\underbrace{\theta_{5,1}}_{0.74982} = \theta_{4,2}\,; \quad \theta_{4,2} = 0.78632,$$

$$i = 5 \Rightarrow 0.5\,\theta_{4,1} + 0.5\,\underbrace{\theta_{6,1}}_{\theta_{4,1}} = \theta_{5,2}$$

$$\Rightarrow \theta_{4,1} = \theta_{5,2} = 0.76964.$$

Iteration 3 : For $j = 2$ in (3.0.27);

$$i = 0 \Rightarrow 0.5\,\underbrace{\theta_{-1,2}}_{\theta 1,2} + 0.5\,\theta_{1,2} = \theta_{0,3}$$

$$\Rightarrow \theta_{1,2} = \theta_{0,3} = 0.93052,$$

$$i = 1 \Rightarrow 0.5\,\underbrace{\theta_{0,2}}_{0.94792} + 0.5\,\underbrace{\theta_{2,2}}_{0.88537} = \theta_{1,3}\,; \quad \theta_{1,3} = 0.91664,$$

$$i = 2 \Rightarrow 0.5\,\underbrace{\theta_{1,2}}_{0.93052} + 0.5\,\underbrace{\theta_{3,2}}_{0.83031} = \theta_{2,3}\,; \quad \theta_{2,3} = 0.88041,$$

$$i = 3 \Rightarrow 0.5\,\underbrace{\theta_{2,2}}_{0.88537} + 0.5\,\underbrace{\theta_{4,2}}_{0.78632} = \theta_{3,3}\,; \quad \theta_{3,3} = 0.83031,$$

$$i = 4 \Rightarrow 0.5\,\underbrace{\theta_{3,2}}_{0.83031} + 0.5\,\underbrace{\theta_{5,2}}_{0.76964} = \theta_{4,3}\,; \quad \theta_{4,3} = 0.79997,$$

$$i = 5 \Rightarrow 0.5\,\theta_{4,2} + 0.5\,\underbrace{\theta_{6,2}}_{\theta_{4,2}} = \theta_{5,3}$$

$$\Rightarrow \theta_{4,2} = \theta_{5,3} = 0.78632.$$

Iteration 4 : For $j = 3$ in (3.0.27).

$$i = 0 \Rightarrow \underbrace{\theta_{-1,3}}_{\theta_{1,3}} + 0.5\,\theta_{1,3} = \theta_{0,4}$$

$$\Rightarrow \theta_{1,3} = \theta_{0,4} = 0.91664,$$

$$i = 1 \Rightarrow 0.5 \underbrace{\theta_{0,3}}_{0.93052} + 0.5 \underbrace{\theta_{2,3}}_{0.88041} = \theta_{1,4}\,; \quad \theta_{1,4} = 0.90546,$$

$$i = 2 \Rightarrow 0.5 \underbrace{\theta_{1,3}}_{0.91664} + 0.5 \underbrace{\theta_{3,3}}_{0.83584} = \theta_{2,4}\,; \quad \theta_{2,4} = 0.87624,$$

$$i = 3 \Rightarrow 0.5 \underbrace{\theta_{2,3}}_{0.88041} + 0.5 \underbrace{\theta_{4,3}}_{0.79997} = \theta_{3,4}\,; \quad \theta_{3,4} = 0.84018,$$

$$i = 4 \Rightarrow 0.5 \underbrace{\theta_{3,3}}_{0.83584} + 0.5 \underbrace{\theta_{5,3}}_{0.78632} = \theta_{4,4}\,; \quad \theta_{4,4} = 0.81108,$$

$$i = 5 \Rightarrow 0.5\,\theta_{4,3} + 0.5 \underbrace{\theta_{6,3}}_{\theta_{4,3}} = \theta_{5,4}$$

$$\Rightarrow \theta_{4,3} = \theta_{5,4} = 0.79997.$$

Iteration 5 : For $j = 4$ in (3.0.27).

$$i = 0 \Rightarrow 0.5\,\underbrace{\theta_{-1,4}}_{\theta_{1,4}} + 0.5\,\theta_{1,4} = \theta_{0,5}$$

$$\Rightarrow \theta_{1,4} = \theta_{0,5} = 0.90546,$$

$$i = 1 \Rightarrow 0.5 \underbrace{\theta_{0,4}}_{0.91664} + 0.5 \underbrace{\theta_{2,4}}_{0.87624} = \theta_{1,5}\,; \quad \theta_{1,5} = 0.89644,$$

$$i = 2 \Rightarrow 0.5 \underbrace{\theta_{1,4}}_{0.90546} + 0.5 \underbrace{\theta_{3,4}}_{0.84018} = \theta_{2,5}\,; \quad \theta_{2,5} = 0.87282,$$

$$i = 3 \Rightarrow 0.5 \underbrace{\theta_{2,4}}_{0.87624} + 0.5 \underbrace{\theta_{4,4}}_{0.81108} = \theta_{3,5}\,; \quad \theta_{3,5} = 0.84366,$$

$$i = 4 \Rightarrow 0.5 \underbrace{\theta_{3,4}}_{0.84018} + 0.5 \underbrace{\theta_{5,4}}_{0.79997} = \theta_{4,5}\,; \quad \theta_{4,5} = 0.82007,$$

$$i = 5 \Rightarrow 0.5\,\theta_{4,4} + 0.5 \underbrace{\theta_{6,4}}_{\theta_{4,4}} = \theta_{5,5}$$

$$\Rightarrow \theta_{4,4} = \theta_{5,5} = 0.81108.$$

As a result, θ values for IBVP given by (3.0.23) are found using the FTCS method.

To solve the viscous Burgers equation, we use

$$u(x_i, t_j) = -\frac{v}{h}\left(\frac{\theta_{i+1,j} - \theta_{i-1,j}}{\theta_{i,j}}\right),\ i = 0, 1, ..., 4,\ j = 0, 1, ..., 5. \tag{3.0.28}$$

The solutions of the viscous Burgers equation calculated by means of (3.0.28) are listed in Table 3.1.

Table 3.1: Comparison of the numerical and classical solutions for the Example 3.1.1 at various times with $v = 1$, $\Delta x = 0.2$, and $\Delta t = 0.02$.

x	t	Classical Solution	FTCS Solution	BTCS Solution
0.2	0.02	0.43509	0.41707	0.44608
	0.04	0.36336	0.33610	0.37272
	0.06	0.30259	0.27333	0.31257
	0.08	0.25136	0.22309	0.26270
	0.1	0.20839	0.18205	0.22110
0.4	0.02	0.74807	0.70197	0.73707
	0.04	0.61840	0.56592	0.61718
	0.06	0.51051	0.45887	0.51772
	0.08	0.42106	0.37250	0.43478
	0.1	0.34698	0.30322	0.36535
0.6	0.02	0.81079	0.73739	0.75683
	0.04	0.66060	0.59646	0.63585
	0.06	0.53897	0.48119	0.53361
	0.08	0.44023	0.38777	0.44754
	0.1	0.35989	0.31262	0.37526
0.8	0.02	0.53757	0.47431	0.47810
	0.04	0.43216	0.38578	0.40298
	0.06	0.34883	0.30951	0.33835
	0.08	0.28248	0.24787	0.28350
	0.1	0.22933	0.19864	0.23726

The IBVP with $v = 1$ is solved by the BTCS method:

$$0.5\,\theta_{i-1,j+1} - 2\,\theta_{i,j+1} + 0.5\,\theta_{i+1,j+1} = -\theta_{i,j}$$

for $i = 0, ..., 5$.

$$\begin{aligned}
i = 0 &\Rightarrow 0.5\,\theta_{-1,j+1} - 2\,\theta_{0,j+1} + 0.5\,\theta_{1,j+1} = -\theta_{0,j} \\
&\Rightarrow -2\,\theta_{0,j+1} + \theta_{1,j+1} = -\theta_{0,j} \\
i = 1 &\Rightarrow 0.5\,\theta_{0,j+1} - 2\,\theta_{1,j+1} + 0.5\,\theta_{2,j+1} = -\theta_{1,j} \\
i = 2 &\Rightarrow 0.5\,\theta_{1,j+1} - 2\,\theta_{2,j+1} + 0.5\,\theta_{3,j+1} = -\theta_{2,j} \\
i = 3 &\Rightarrow 0.5\,\theta_{2,j+1} - 2\,\theta_{3,j+1} + 0.5\,\theta_{4,j+1} = -\theta_{3,j} \\
i = 4 &\Rightarrow 0.5\,\theta_{3,j+1} - 2\,\theta_{4,j+1} + 0.5\,\theta_{5,j+1} = -\theta_{4,j} \\
i = 5 &\Rightarrow 0.5\,\theta_{4,j+1} - 2\,\theta_{5,j+1} + 0.5\,\theta_{6,j+1} = -\theta_{5,j} \\
&\Rightarrow \theta_{4,j+1} + -2\,\theta_{5,j+1} = -\theta_{6,j}.
\end{aligned}$$

The matrix product shown below can be written to demonstrate the equation system above.

$$\underbrace{\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 0.5 & -2 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & -2 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & -2 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & -2 & 0.5 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}}_{A} \begin{bmatrix} \theta_{0,j+1} \\ \theta_{1,j+1} \\ \theta_{2,j+1} \\ \theta_{3,j+1} \\ \theta_{4,j+1} \\ \theta_{5,j+1} \end{bmatrix} = \begin{bmatrix} -\theta_{0,j} \\ -\theta_{1,j} \\ -\theta_{2,j} \\ -\theta_{3,j} \\ -\theta_{4,j} \\ -\theta_{5,j} \end{bmatrix} \tag{3.0.29}$$

The LU decomposition method is used to solve matrix equation (3.0.29). In order to do this, the matrix A is decomposed as $[L]$ lower triangular matrix and $[U]$ upper triangular matrix. That is, $[A] = [L][U]$ is satisfied. The $[L]$ and $[U]$ matrices are obtained after the relevant calculations are completed.

$$[L] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.25 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.28571 & 1 & 0 & 0 & 0 \\ 0 & 0 & -0.26923 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.26804 & 1 & 0 \\ 0 & 0 & 0 & 0 & -0.53591 & 1 \end{bmatrix}$$

and

$$[U] = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1.75 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & -1.85714 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & -1.86538 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & -1.86597 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & -1.73204 \end{bmatrix}$$

Iteration 1 : (3.0.29) is calculated for $j = 0$. The matrix $[X]$ is obtained using the equation $[L][X] = [\theta_{i,0}]$ for $i = 0, 1, ..., 5$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.25 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.28571 & 1 & 0 & 0 & 0 \\ 0 & 0 & -0.26923 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.26804 & 1 & 0 \\ 0 & 0 & 0 & 0 & -0.53591 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ -0.97006 \\ -0.89585 \\ -0.81193 \\ -0.74982 \\ -0.72737 \end{bmatrix}.$$

The matrix $[X]$ is determined by solving the aforementioned system of equations:

$$[X] = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ -1.22006 \\ -1.24443 \\ -1.14696 \\ -1.05725 \\ -1.29396 \end{bmatrix}$$

The matrix $[\theta_{i,1}]$ is found using the equation $[U][\theta_{i,1}] = [X]$ for $i = 0, 1, ..., 5$.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1.75 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & -1.85714 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & -1.86538 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & -1.86597 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & -1.73204 \end{bmatrix} \begin{bmatrix} \theta_{0,1} \\ \theta_{1,1} \\ \theta_{2,1} \\ \theta_{3,1} \\ \theta_{4,1} \\ \theta_{5,1} \end{bmatrix} = \begin{bmatrix} -1 \\ -1.22006 \\ -1.24443 \\ -1.14696 \\ -1.05725 \\ -1.29396 \end{bmatrix}.$$

As a result of the calculations, the matrix $[\theta_{i,1}]$

$$[\theta_{i,1}] = \begin{bmatrix} \theta_{0,1} \\ \theta_{1,1} \\ \theta_{2,1} \\ \theta_{3,1} \\ \theta_{4,1} \\ \theta_{5,1} \end{bmatrix} = \begin{bmatrix} 0.97586 \\ 0.95173 \\ 0.89095 \\ 0.82039 \\ 0.76677 \\ 0.74707 \end{bmatrix}$$

is generated.

Iteration 2 : (3.0.29) is calculated for $j = 1$. The matrix $[X]$ is obtained using the equation $[L][X] = [\theta_{i,1}]$ for $i = 0, 1, ..., 5$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.28571 & 1 & 0 & 0 & 0 \\ 0 & 0 & -0.26923 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.26804 & 1 & 0 \\ 0 & 0 & 0 & 0 & -0.53591 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -0.97586 \\ -0.95173 \\ -0.89095 \\ -0.82039 \\ -0.76677 \\ -0.74707 \end{bmatrix},$$

$$[X] = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -0.97586 \\ -1.19569 \\ -1.23257 \\ -1.15223 \\ -1.07561 \\ -1.32350 \end{bmatrix}.$$

The matrix $[\theta_{i,1}]$ is obtained using the equation $[U][\theta_{i,2}] = [X]$ for $i = 0, 1, ..., 5$.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1.75 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & -1.85714 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & -1.86538 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & -1.86597 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & -1.73204 \end{bmatrix} \begin{bmatrix} \theta_{0,2} \\ \theta_{1,2} \\ \theta_{2,2} \\ \theta_{3,2} \\ \theta_{4,2} \\ \theta_{5,2} \end{bmatrix} = \begin{bmatrix} -0.97586 \\ -1.19569 \\ -1.23257 \\ -1.15223 \\ -1.07561 \\ -1.32350 \end{bmatrix},$$

$$[\theta_{i,2}] = \begin{bmatrix} \theta_{0,2} \\ \theta_{1,2} \\ \theta_{2,2} \\ \theta_{3,2} \\ \theta_{4,2} \\ \theta_{5,2} \end{bmatrix} = \begin{bmatrix} 0.95617 \\ 0.93649 \\ 0.88636 \\ 0.82708 \\ 0.78118 \\ 0.76412 \end{bmatrix}.$$

Iteration 3 : (3.0.29) is calculated for $j = 2$. The matrix $[X]$ is obtained using the equation $[L][X] = [\theta_{i,2}]$ for $i = 0, 1, ..., 5$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.28571 & 1 & 0 & 0 & 0 \\ 0 & 0 & -0.26923 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.26804 & 1 & 0 \\ 0 & 0 & 0 & 0 & -0.53591 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -0.95617 \\ -0.93649 \\ -0.88636 \\ -0.82708 \\ -0.78118 \\ -0.76412 \end{bmatrix},$$

$$[X] = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -0.95617 \\ -1.17553 \\ -1.22222 \\ -1.15613 \\ -1.09107 \\ -1.34883 \end{bmatrix}.$$

The matrix $[\theta_{i,2}]$ is obtained using the equation $[U][\theta_{i,3}] = [X]$ for $i = 0, 1, ..., 5$.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1.75 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & -1.85714 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & -1.86538 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & -1.86597 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & -1.73204 \end{bmatrix} \begin{bmatrix} \theta_{0,3} \\ \theta_{1,3} \\ \theta_{2,3} \\ \theta_{3,3} \\ \theta_{4,3} \\ \theta_{5,3} \end{bmatrix} = \begin{bmatrix} -0.95617 \\ -1.17553 \\ -1.22222 \\ -1.15613 \\ -1.09107 \\ -1.34883 \end{bmatrix},$$

$$[\theta_{i,3}] = \begin{bmatrix} \theta_{0,3} \\ \theta_{1,3} \\ \theta_{2,3} \\ \theta_{3,3} \\ \theta_{4,3} \\ \theta_{5,3} \end{bmatrix} = \begin{bmatrix} 0.93998 \\ 0.92379 \\ 0.88223 \\ 0.83244 \\ 0.79339 \\ 0.77875 \end{bmatrix}.$$

Iteration 4 : (3.0.29) is calculated for $j = 3$. The matrix $[X]$ is obtained using the equation $[L][X] = [\theta_{i,3}]$ for $i = 0, 1, ..., 5$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.28571 & 1 & 0 & 0 & 0 \\ 0 & 0 & -0.26923 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.26804 & 1 & 0 \\ 0 & 0 & 0 & 0 & -0.53591 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -0.93998 \\ -0.92379 \\ -0.88223 \\ -0.83244 \\ -0.79339 \\ -0.77875 \end{bmatrix},$$

$$[X] = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -0.93998 \\ -1.15878 \\ -1.21330 \\ -1.15909 \\ -1.10407 \\ -1.37043 \end{bmatrix}.$$

The matrix $[\theta_{i,3}]$ is obtained using the equation $[U][\theta_{i,4}] = [X]$ for $i = 0, 1, ..., 5$.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1.75 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & -1.85714 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & -1.86538 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & -1.86597 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & -1.73204 \end{bmatrix} \begin{bmatrix} \theta_{0,4} \\ \theta_{1,4} \\ \theta_{2,4} \\ \theta_{3,4} \\ \theta_{4,4} \\ \theta_{5,4} \end{bmatrix} = \begin{bmatrix} -0.93998 \\ -1.15878 \\ -1.21330 \\ -1.15909 \\ -1.10407 \\ -1.37043 \end{bmatrix},$$

$$[\theta_{i,4}] = \begin{bmatrix} \theta_{0,2} \\ \theta_{1,2} \\ \theta_{2,2} \\ \theta_{3,2} \\ \theta_{4,2} \\ \theta_{5,2} \end{bmatrix} = \begin{bmatrix} 0.92658 \\ 0.91319 \\ 0.87860 \\ 0.83679 \\ 0.80370 \\ 0.79122 \end{bmatrix}.$$

Iteration 5 : (3.0.29) is calculated for $j = 4$. The matrix $[X]$ is obtained using the equation $[L][X] = [\theta_{i,4}]$ for $i = 0, 1, ..., 5$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.28571 & 1 & 0 & 0 & 0 \\ 0 & 0 & -0.26923 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.26804 & 1 & 0 \\ 0 & 0 & 0 & 0 & -0.53591 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -0.92658 \\ -0.91319 \\ -0.87860 \\ -0.83679 \\ -0.80370 \\ -0.79122 \end{bmatrix},$$

$$[X] = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -0.92658 \\ -1.14483 \\ -1.20569 \\ -1.16139 \\ -1.11500 \\ -1.38876 \end{bmatrix}.$$

The matrix $[\theta_{i,4}]$ is obtained using the equation $[U][\theta_{i,5}] = [X]$ for $i = 0, 1, ..., 5$.

$$\begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1.75 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & -1.85714 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & -1.86538 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & -1.86597 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & -1.73204 \end{bmatrix} \begin{bmatrix} \theta_{0,5} \\ \theta_{1,5} \\ \theta_{2,5} \\ \theta_{3,5} \\ \theta_{4,5} \\ \theta_{5,5} \end{bmatrix} = \begin{bmatrix} -0.92658 \\ -1.14483 \\ -1.20569 \\ -1.16139 \\ -1.11500 \\ -1.38876 \end{bmatrix},$$

$$[\theta_{i,5}] = \begin{bmatrix} \theta_{0,5} \\ \theta_{1,5} \\ \theta_{2,5} \\ \theta_{3,5} \\ \theta_{4,5} \\ \theta_{5,5} \end{bmatrix} = \begin{bmatrix} 0.91545 \\ 0.90432 \\ 0.87546 \\ 0.84035 \\ 0.81239 \\ 0.80180 \end{bmatrix}.$$

So far, the BTCS method has been used to solve the IBVP (3.0.23). The values determined are now substituted in the equation (3.0.28) to obtain the numerical solutions to the problem (3.0.22). As a result, the solutions in Table 3.1 are expressed.

Table 3.2: Comparison of the absolute error and classical solution for the Example 3.1.1 at various times with $v = 1$, $\Delta x = 0.2$, and $\Delta t = 0.02$.

x	t	Classical Solution	Absolute Error for FTCS	Absolute Error for BTCS
0.2	0.02	0.43509	0.01802	0.01099
	0.04	0.36336	0.02726	0.00936
	0.06	0.30259	0.02926	0.00998
	0.08	0.25136	0.02827	0.01134
	0.1	0.20839	0.02634	0.01271
0.4	0.02	0.74807	0.0461	0.011
	0.04	0.61840	0.05248	0.00122
	0.06	0.51051	0.05164	0.00721
	0.08	0.42106	0.04856	0.01372
	0.1	0.34698	0.04376	0.01837
0.6	0.02	0.81079	0.0734	0.05396
	0.04	0.66060	0.06414	0.02475
	0.06	0.53897	0.05778	0.00536
	0.08	0.44023	0.05246	0.00731
	0.1	0.35989	0.04727	0.01537
0.8	0.02	0.53757	0.06326	0.05947
	0.04	0.43216	0.04638	0.02918
	0.06	0.34883	0.03932	0.01048
	0.08	0.28248	0.03461	0.00102
	0.1	0.22933	0.03069	0.00803

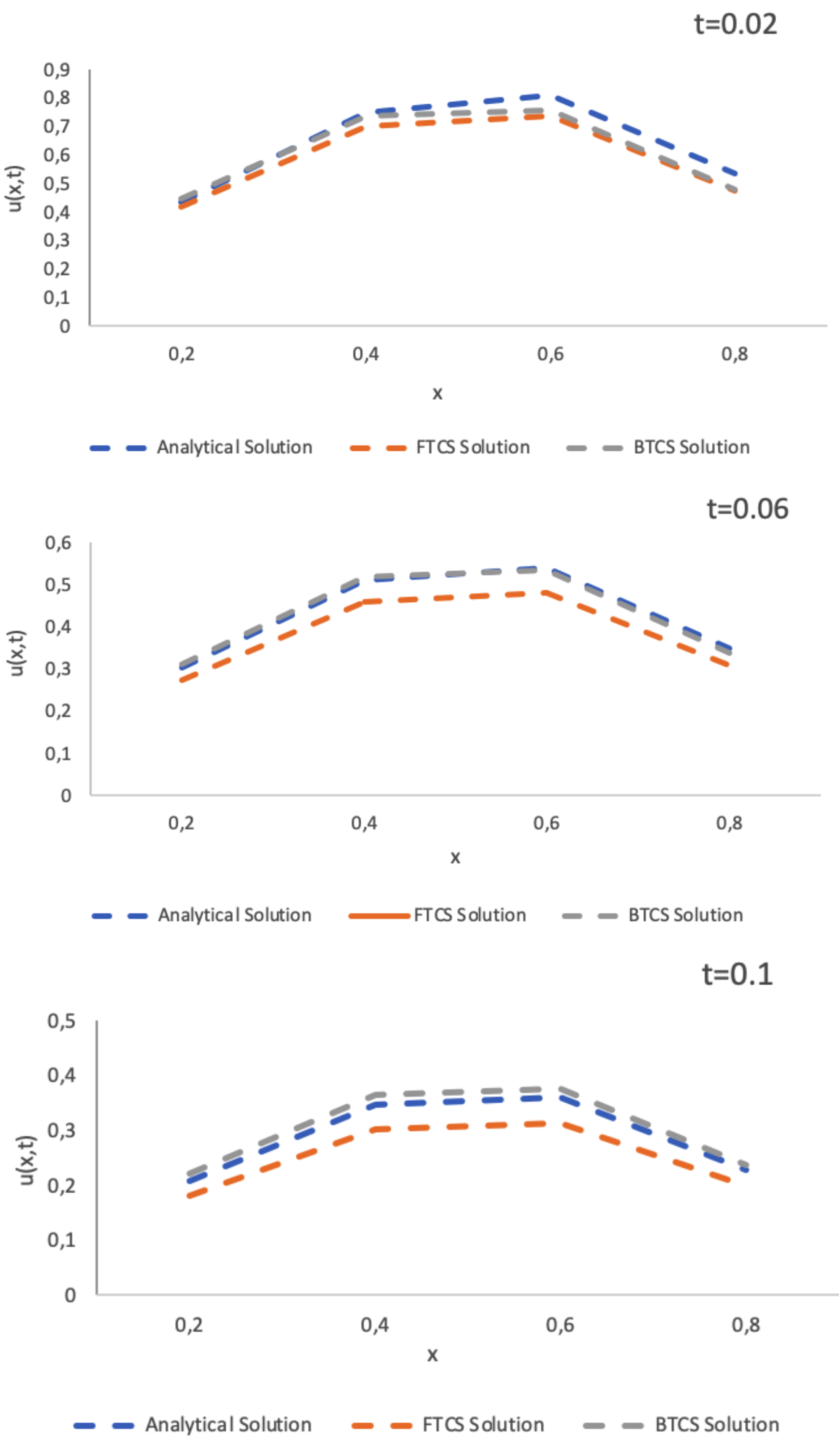

Figure 3.8: Solutions for $v = 1$ at different times t in **Example 3.0.1**

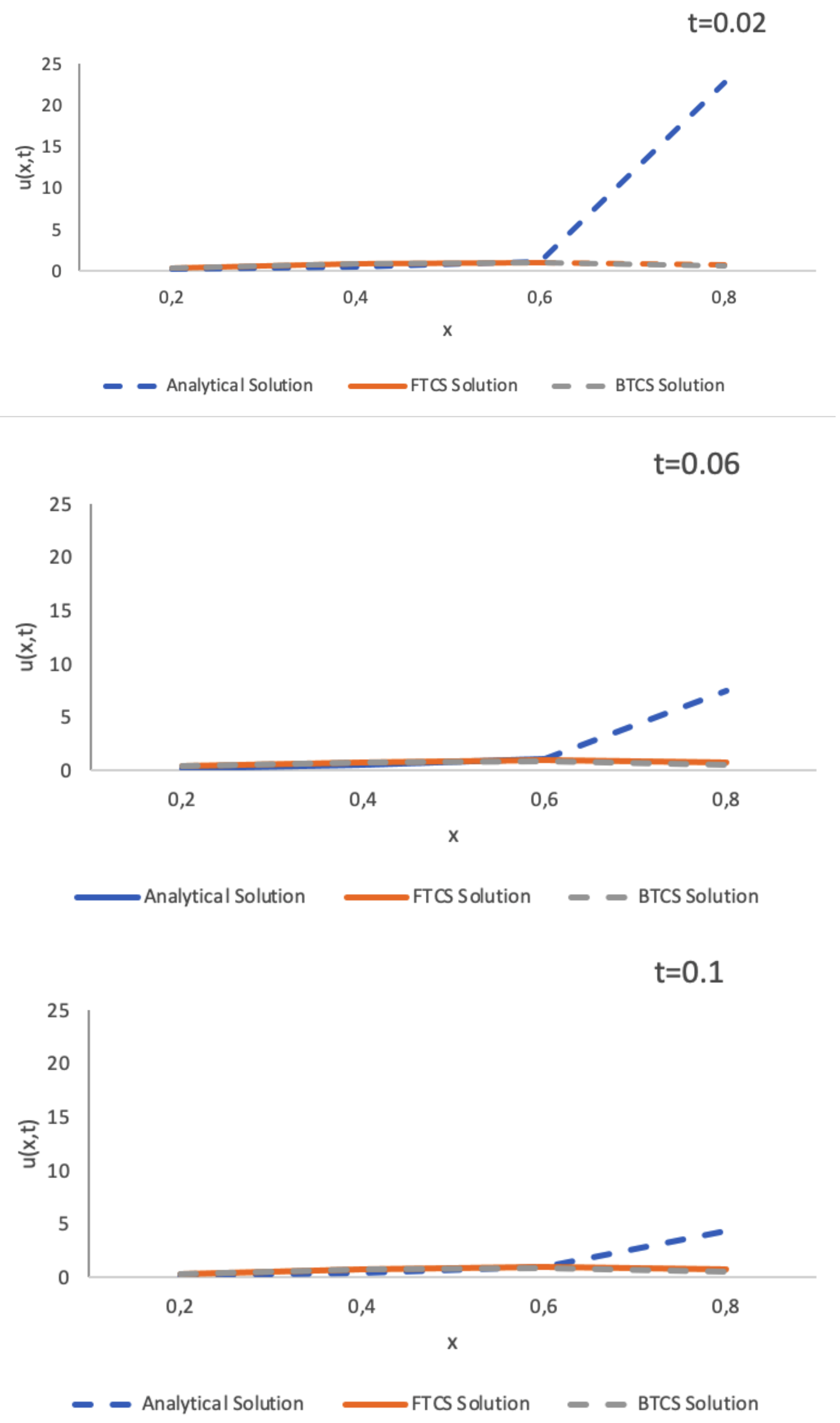

Figure 3.9: Solutions for $v = 0.1$ at different times t in **Example 3.0.1**

Table 3.3: Comparison of the numerical and classical solutions for the Example 3.1.1 at various times with $v = 0.1$, $\Delta x = 0.2$, and $\Delta t = 0.02$.

x	t	Classical Solution	FTCS Solution	BTCS Solution
0.2	0.02	0.22598	0.43176	0.43358
	0.04	0.22373	0.41320	0.41647
	0.06	0.22148	0.39609	0.40050
	0.08	0.21923	0.38030	0.38558
	0.1	0.21698	0.36567	0.37158
0.4	0.02	0.52687	0.87412	0.87605
	0.04	0.51930	0.83202	0.83502
	0.06	0.51184	0.79404	0.79739
	0.08	0.50444	0.75960	0.76266
	0.1	0.49711	0.72814	0.73044
0.6	0.02	1.15759	1.07962	1.05137
	0.04	1.12186	1.04732	0.99381
	0.06	1.08762	1.01535	0.93893
	0.08	1.05473	0.98393	0.88677
	0.1	1.02323	0.95337	0.83733
0.8	0.02	22.75834	0.75259	0.67771
	0.04	11.30274	0.76147	0.62765
	0.06	7.46747	0.76585	0.58535
	0.08	5.07653	0.76645	0.54832
	0.1	4.39526	0.76391	0.51529

Table 3.4: Comparison of the absolute error and classical solution for the Example 3.1.1 at various times with $v = 0.1$, $\Delta x = 0.2$, and $\Delta t = 0.02$.

x	t	Classical Solution	Absolute Error for FTCS	Absolute Error for BTCS
0.2	0.02	0.22598	0.20578	0.2076
	0.04	0.22373	0.18947	0.19274
	0.06	0.22148	0.17461	0.17902
	0.08	0.21923	0.16107	0.16635
	0.1	0.21698	0.14869	0.1546
0.4	0.02	0.52687	0.34725	0.34918
	0.04	0.51930	0.31272	0.31572
	0.06	0.51184	0.2822	0.28555
	0.08	0.50444	0.25516	0.25822
	0.1	0.49711	0.23103	0.23333
0.6	0.02	1.15759	0.07797	0.10622
	0.04	1.12186	0.07454	0.12805
	0.06	1.08762	0.07227	0.14869
	0.08	1.05473	0.0708	0.16796
	0.1	1.02323	0.06986	0.1859
0.8	0.02	22.75834	22.00575	22.08063
	0.04	11.30274	10,54127	10.67509
	0.06	7.46747	6.70162	6.88212
	0.08	5.07653	4.31008	4.52821
	0.1	4.39526	3.63135	3.87997

Example 3.0.2. Consider *Example 2.4.2*. The classical and numerical solutions for this problem are given in the tables.

Table 3.5: Comparison of the numerical and classical solutions for the Example 3.1.2 at various times with $v = 1$, $\Delta x = 0.2$, and $\Delta t = 0.02$.

x	t	Classical Solution	FTCS Solution	BTCS Solution
0.2	0.02	0.23589	0.21924	0.23831
	0.04	0.19544	0.18039	0.19796
	0.06	0.16168	0.14510	0.16536
	0.08	0.13356	0.11845	0.13865
	0.1	0.11021	0.09603	0.11640
0.4	0.02	0.39428	0.36885	0.38057
	0.04	0.32484	0.29557	0.32017
	0.06	0.26745	0.24041	0.26912
	0.08	0.22009	0.19432	0.22615
	0.1	0.18103	0.15782	0.19007
0.6	0.02	0.41104	0.37776	0.38546
	0.04	0.33613	0.30409	0.32493
	0.06	0.27506	0.24596	0.27326
	0.08	0.22521	0.19875	0.22955
	0.1	0.18448	0.16024	0.19271
0.8	0.02	0.26308	0.23433	0.24665
	0.04	0.21375	0.19320	0.20587
	0.06	0.17400	0.15493	0.17216
	0.08	0.14186	0.12484	0.14412
	0.1	0.11580	0.10058	0.12069

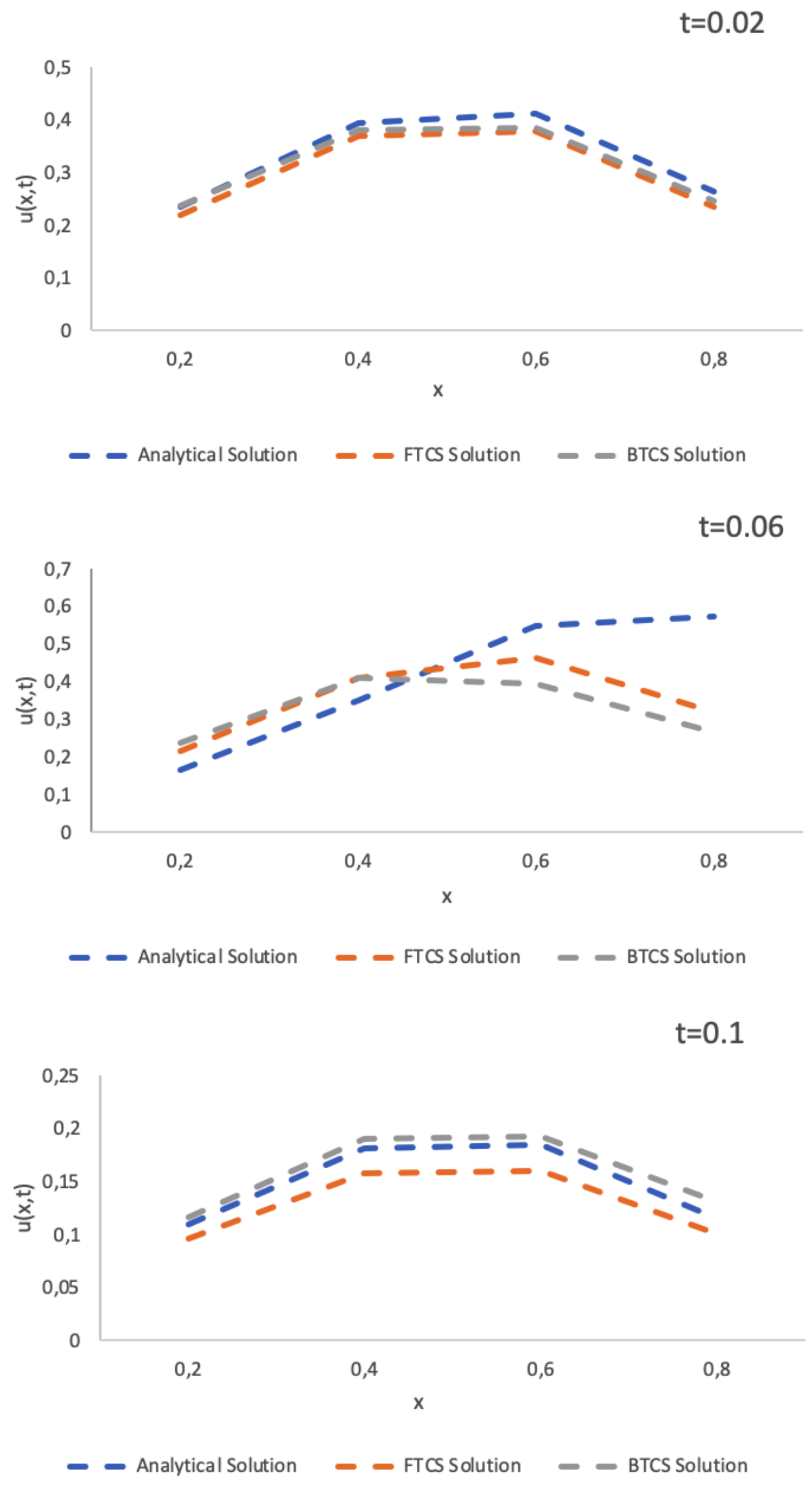

Figure 3.10: Solutions for $v = 1$ at different times t in **Example 3.0.2**

Table 3.6: Comparison of the absolute error and classical solution for the Example 3.1.2 at various times with $v = 1$, $\Delta x = 0.2$, and $\Delta t = 0.02$.

x	t	Classical Solution	Absolute Error for FTCS	Absolute Error for BTCS
0.2	0.02	0.23589	0.01665	0.00242
	0.04	0.19544	0.01505	0.00252
	0.06	0.16168	0.01658	0.00368
	0.08	0.13356	0.01511	0.00509
	0.1	0.11021	0.01418	0.00619
0.4	0.02	0.39428	0.02543	0.01371
	0.04	0.32484	0.02927	0.00467
	0.06	0.26745	0.02704	0.00167
	0.08	0.22009	0.02577	0.00606
	0.1	0.18103	0.02321	0.00904
0.6	0.02	0.41104	0.03328	0.02558
	0.04	0.33613	0.03204	0.0112
	0.06	0.27506	0.0291	0.0018
	0.08	0.22521	0.02646	0.00434
	0.1	0.18448	0.02424	0.00823
0.8	0.02	0.26308	0.02875	0.01643
	0.04	0.21375	0.02055	0.00788
	0.06	0.17400	0.01907	0.00184
	0.08	0.14186	0.01702	0.00226
	0.1	0.11580	0.01522	0.00489

Table 3.7: Comparison of the numerical and classical solutions for the Example 3.1.2 at various times with $v = 0.1$, $\Delta x = 0.2$, and $\Delta t = 0.02$.

x	t	Classical Solution	FTCS Solution	BTCS Solution
0.2	0.02	0.17165	0.22915	0.25505
	0.04	0.16954	0.22315	0.24669
	0.06	0.16744	0.21735	0.23878
	0.08	0.16535	0.21175	0.23126
	0.1	0.16327	0.20634	0.22409
0.4	0.02	0.36182	0.43775	0.43753
	0.04	0.35605	0.42359	0.42440
	0.06	0.35035	0.41046	0.41130
	0.08	0.34472	0.39819	0.39828
	0.1	0.33916	0.38671	0.38533
0.6	0.02	0.57818	0.48018	0.45608
	0.04	0.56360	0.47199	0.42542
	0.06	0.54944	0.46351	0.39616
	0.08	0.53572	0.45485	0.36823
	0.1	0.52242	0.44611	0.34158
0.8	0.02	0.69223	0.33299	0.30967
	0.04	0.65917	0.33012	0.28728
	0.06	0.62855	0.32693	0.26736
	0.08	0.60010	0.32343	0.24937
	0.1	0.57364	0.31964	0.23292

The tables provide the numerical outcomes for the values of $v = 1$ and $v = 0.1$. When the outcomes are compared, it is seen that the numerical results for $v = 0.1$ tend to decrease.

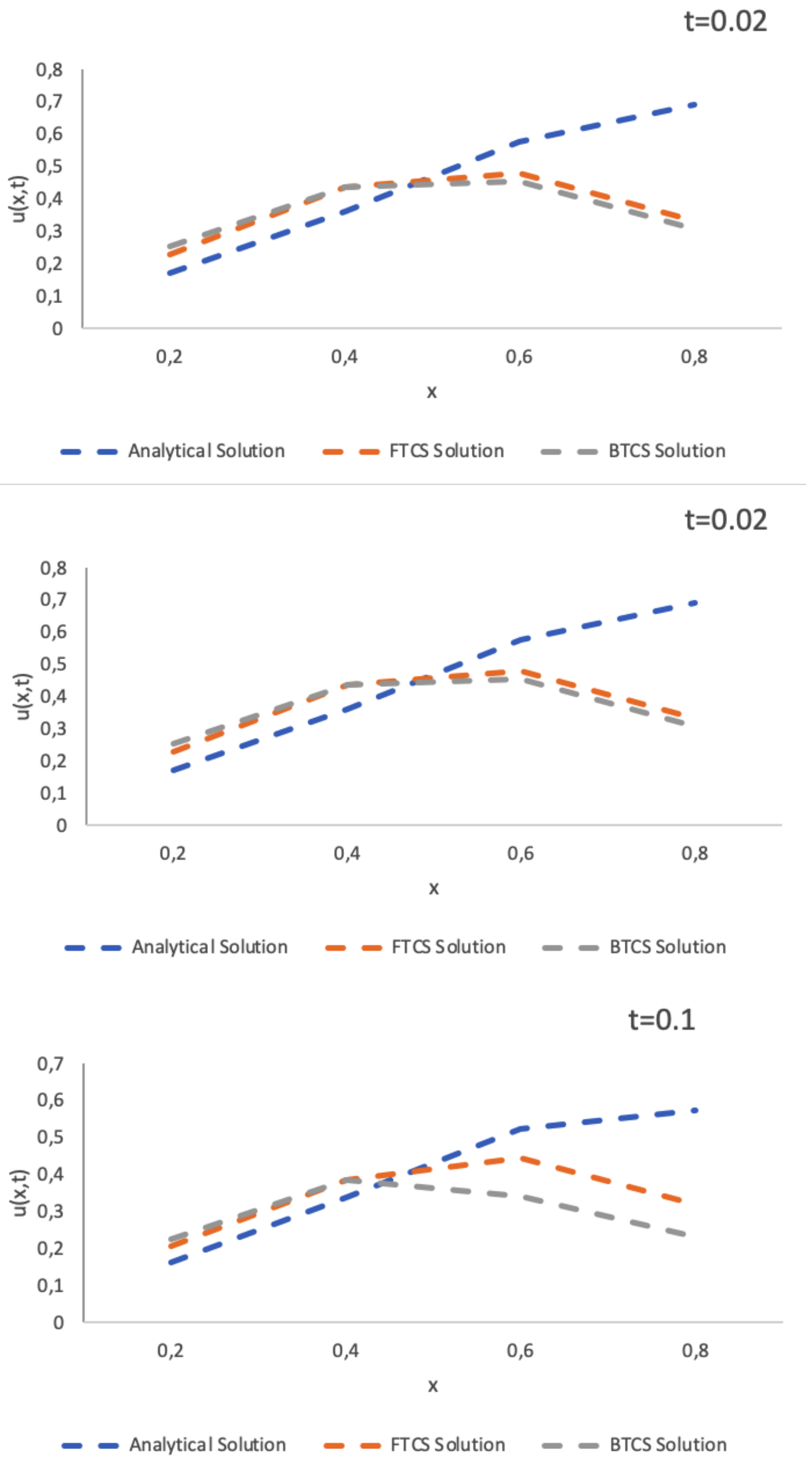

Figure 3.11: Solutions for $v = 0.1$ at different times t in **Example 3.0.2**

Table 3.8: Comparison of the absolute error and classical solution for the Example 3.1.2 at various times with $v = 0.1$, $\Delta x = 0.2$, and $\Delta t = 0.02$.

x	t	Classical Solution	Absolute Error for FTCS	Absolute Error for BTCS
0.2	0.02	0.17165	0.0575	0.0834
	0.04	0.16954	0.05361	0.07715
	0.06	0.16744	0.04991	0.07134
	0.08	0.16535	0.0464	0.06591
	0.1	0.16327	0.04307	0.06082
0.4	0.02	0.36182	0.07593	0.07571
	0.04	0.35605	0.06754	0.06835
	0.06	0.35035	0.06011	0.06095
	0.08	0.34472	0.05347	0.05356
	0.1	0.33916	0.04755	0.04617
0.6	0.02	0.57818	0.098	0.1221
	0.04	0.56360	0.09161	0.13818
	0.06	0.54944	0.08593	0.15328
	0.08	0.53572	0.08087	0.16749
	0.1	0.52242	0.07631	0.18084
0.8	0.02	0.69223	0.35924	0.38256
	0.04	0.65917	0.32905	0.37189
	0.06	0.62855	0.30162	0.36119
	0.08	0.60010	0.27667	0.35073
	0.1	0.57364	0.254	0.34072

CHAPTER 4

CONCLUSION

In this thesis, the classical and numerical solutions of viscous Burgers equation are studied. Burgers equations, with and without viscosity, are introduced in the first section with some basic information. This equation, which is a quasilinear PDE, can be analytically solved using a variety of techniques. In the second part, the viscous Burgers equation is solved by applying the Hopf-Cole transformation. The non-linear Burgers equation is transformed into the linear PDE heat equation as a result of this transformation. Using the method of separation of variables and the Fourier transform, the derived heat equation is solved. This method is implemented to solve three IBVPs which is presented with various initial condition. In the third chapter, the viscous Burgers equation is numerically solved by using FDM. BTCS is applied to the IBVPs, which is an implicit numerical method, and FTCS, which is an explicit numerical method, and the outcomes are provided in tables. Problems solved in previous studies in the literature are discussed for large domains. IBVPs defined on a smaller domain are solved in this study, comparing to studies in the literature. The outcomes of the FTCS and BTCS methods are analyzed and are shown graphically. This work demonstrated that the FTCS and BTCS methods performed well in smaller domains, and the BTCS method yielded outcomes that are reasonably similar to the analytical solution. As a result, it is observed that the numerical methods used in this study provides reasonable results when compared with analytical solutions.

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